Department of Mathematics
Unit II - Fourier Transforms
Fourier Transform and its inverse

UNIT-2 FOURIER TRANSFORM

The fourier transform of a function is $F(s) = F[f(x)]$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

1. Prove that the fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a \end{cases} \text{ is } 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$

and deduce $\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$ and also using Parseval's identity. Prove that

$$\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos 3x + i\sin 3x)\, dx$$

$$[\because e^{isx} = \cos ix + i\sin sx]$$

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$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a^2-x^2)\cos x \, dx + i\int_{-a}^{a} (a^2-x^2) \sin sx \, dx \right\}$$

[∵ It is odd function odd = 0]

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2-x^2)\cos sx \, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \times 2\int_{0}^{a} (a^2-x^2)\cos sx \, dx$$

[∵ even function $2\int_0^a f(x)$]

$u = (a^2-x^2)$ $\quad dv = \cos sx \, dx$

$u' = -2x$ $\quad v = \frac{\sin sx}{s}$

$u'' = -2$ $\quad v_1 = \frac{-\cos sx}{s^2}$

$u''' = 0$ $\quad v_2 = -\frac{\sin sx}{s^3}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \times 2 \left\{ \left[(a^2-x^2)\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + \frac{2\sin sx}{s^3} \right]_0^a \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \left\{ \left[0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3} \right] - [0-0+0] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \left[\frac{-2as\cos as + 2\sin as}{s^3} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - as\cos as}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$

Inverse,

(i) $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right](\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds - i\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\sin sx\, ds\right\}$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin as - as\cos as}{s^3}\right]\cos sx\, ds$$

$$f(x) = \frac{2}{\pi}\times 2\int_{0}^{\infty}\left[\frac{\sin as - as\cos as}{s^3}\right]\cos sx\, ds$$

Sub $x = 0$, $a = 1$ $\Rightarrow$

$$f(0) = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos 0\, ds$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\, ds \Rightarrow \frac{\pi}{4} = \int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\, ds$$

$$\left[\begin{array}{l} \because f(x) = a^2 - x^2 \\ f(0) = a^2 \\ f(0) = 1 \ [\because a = 1] \end{array}\right]$$

Replace 's' = 't' $\Rightarrow$

$$\frac{\pi}{4} = \int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt$$

(ii) Using Parsaval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} (f(x))^2 dx$$

$$\int_{-\infty}^{\infty} \left[\frac{2\sqrt{2}}{\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-a}^{a} (a^2 - x^2)^2 dx$$

Sub $a = 1 \Rightarrow$

$$\frac{4\times 2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = 2\int_{0}^{1} (1^2 - x^2)^2 dx$$

$$\frac{8}{\pi}\times 2\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = 2\int_{0}^{1} x\left[1 - 2x^2 + x^4\right) dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= 2\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{15 - 10 + 3}{15}$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\cancel{16}}{15}\times\frac{\pi}{\cancel{16}}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

Replace 's' by 't' $\Rightarrow$

$$\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}$$

(2) Find the fourier transform of

$$f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a \end{cases}$$ and hence deduce

the interval $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ and Prove

that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$.

Solution:

We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\cos sx\, dx + i\int_{-a}^{a} (a-|x|)\sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 2\int_0^a [a-|x|\cos sx\, dx + 0$$

$[\because \text{odd} = 0]$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a a\cos sx\, dx - \int_0^a x\cos sx\, dx$$

$u = (a-x) \qquad dv = \cos sx\, dx$

$u' = -1 \qquad v = \frac{\sin sx}{s}$

$u'' = 0 \qquad v_1 = \frac{-\cos sx}{s^2}$

$$= \frac{2}{\sqrt{2\pi}}\left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left(0 - \frac{\cos sa}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{\cos as}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1-\cos as}{s^2}\right]$$

$\left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right.$

$2\sin^2 x = 1-\cos 2x$

$\left.2\sin^2\left(\frac{x}{2}\right) = 1-\cos x\right]$

$$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\right]$$

Inverse

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\, \cos isx - i\sin sx\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\, \cos sx\, ds - i\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}}\, \frac{2\sin^2\left(\frac{as}{2}\right)}{s^2}\, \sin sx\, ds$$

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$$= \frac{1}{\sqrt{2\pi}} \times \frac{\sqrt{2}}{\sqrt{\pi}} \times 2 \times 2 \int_0^\infty \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \cos sx \, ds$$

$$F(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \cos sx \, ds$$

Sub $x = 0$, $a = 2$

$$F(0) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 s}{s^2} \, ds$$

Sub 's' by 't'

$$2 = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 t}{t^2} \, dt \qquad \left[\begin{array}{l} F(x) = a - x \\ F(0) = 2 - 0 \\ F(0) = 2 \end{array} \right]$$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin^2 t}{dt}$$

Using Parseval Identity

$$\int_{-\infty}^\infty |F(s)| \, ds = \int_{-a}^a |f(x)|^2 dx$$

$$2 \int_0^\infty \left[\frac{\sqrt{2}}{\sqrt{\pi}} \frac{2 \sin^2\left(\frac{as}{2}\right)}{s^2} \right]^2 ds = 2 \int_0^a (a - x)^2 \, dx$$

$$\frac{4}{\pi} \times 4 \int_0^\infty \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \, ds = 2 \int_0^a (a^2 + x^2 - 2ax) \, dx$$

Sub $a = 2$

$$\frac{16}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = 2\int_0^a (4+x^2-4x)\,dx$$

$$\frac{16}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = 2\left[4x + \frac{x^3}{3} + \frac{2}{2}x^2\right]_0^2$$

$$= 2\left[8 + \frac{8}{3} - 8\right]$$

$$\frac{\cancel{16}}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = \frac{\cancel{16}}{\cancel{2}\,3}$$

Replace 's' by 't'

$$\left(\frac{\sin^2 t}{t^2}\right)^2 dt = \frac{\pi}{3}$$