UNIT-2 Fourier Transform

(X)

Fourier Transform :→ (FT)

FT for $f(x)$ is defined by $F[f(x)]$

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inversion Formula :→

$$F^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Parseval's Identity :→

$$\int_{-\infty}^{\infty} [f(x)]^2\, dx = \int_{-\infty}^{\infty} [F(s)]^2\, ds$$

NOTE :→

* $F(s)$ and $F^{-1}[F(s)]$ is called Fourier Transform pair

* $e^{i\theta} = \cos\theta + i\sin\theta$

① Find the fourier Transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

and deduce that $\int_0^{\infty} \frac{\sin t}{t}\, dt$ & $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$

$$f(x) = \begin{cases} 1, & |x| < a \quad -a < x < a \\ 0, & |x| > a \quad -\infty < x < -a \\ & \quad +a < x < \infty \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 + \int_{-a}^{a} e^{isx} dx + \int_{a}^{\infty} 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} \overset{\text{even}}{\cos sx} + i \overset{\text{odd}}{\sin sx}\, dx \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{\cancel{\cos s}\ \sin sx}{s} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \int_{0}^{a} \cos sx \, dx \quad = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_{0}^{a}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$

$$F^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} e^{-isx} ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \sqrt{2/\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} [\cos sx - i \sin sx] ds$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx - i \frac{\sin as}{s} \sin sx \quad ds$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\overset{\text{odd}}{\sin as}}{\underset{\text{odd}}{s}} \overset{\text{even}}{\cos sx} ds - \frac{1}{\pi} \int_{-\infty}^{\infty} i \frac{\overset{\text{even}}{\sin as}}{\underset{\text{odd}}{s}} \overset{\text{odd}}{\sin sx}\, ds \underset{=0}{}$$

$$1 = \frac{1}{\pi} \left[2 \int_{0}^{\infty} \frac{\sin as}{s} \cos sx \, ds \right] - 0$$

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$$1 = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty \frac{\sin as}{s} \cos sx\, ds\right.$$

Put $s = t$ $\quad a = 1$ $\quad x = 0$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin t}{t}\, dt$$

$$F(s) = \sqrt{\frac{2}{\pi}}\, \frac{\sin sa}{s}$$

By using $\int_{-\infty}^{\infty} (f(x))^2 dx = \int_{-\infty}^{\infty} (F(s))^2 ds$

$$\cancel{\int_{-\infty}^{-a} (f(x))^2 dx} + \int_{-a}^{a} (f(x))^2 dx + \cancel{\int_a^\infty (f(x))^2 dx} = \int_{-\infty}^{\infty} (F(s))^2 ds$$

$$\int_{-a}^{a} 1^2 dx = \int_{-\infty}^{\infty} (F(s))^2 ds$$

even

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin sa)^2}{s^2} ds$$

$$2a = \frac{2}{\pi} \times 2 \int_0^\infty \frac{\sin^2 sa}{s^2} ds$$

Put $s = t$ $\quad a = 1$

$$\cancel{2}(1) = \frac{\cancel{4}^{2}}{\pi}\int_0^\infty \frac{\sin^2 t}{t^2} dt$$

$$\frac{\pi}{2} = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$$

2) Find the Fourier Transform for

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$ also deduct

that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ & $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

given

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \Rightarrow -1<|x|<1 \\ 0, & |x|>1 \quad -\infty<|x|<-1 \\ & \quad 1<|x|<\infty \end{cases}$$

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} \cdot \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} f(x) e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x) e^{isx} dx + \int_1^\infty f(x) e^{isx} dx$$

since 0

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) (\cos sx + i \sin sx)\, dx$$

$f(x) = 1-|x|$

$= 1-|-x|$

$= 1-x$ is even function

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underset{\text{even}}{(1-|x|)} \underset{\text{even}}{\cos sx} + \cancel{i\, \underset{\text{even}}{(1-|x|)}\, \underset{\text{odd}}{\sin sx}}\; dx$$

since odd

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \int_0^1 (1-|x|) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^1 (1-|x|) \cos sx \, dx$$

$u = 1-|x|$ $\quad v = \cos sx$

$u' = -1$ $\quad v_1 = \frac{\sin sx}{s}$ $\quad v_2 = \frac{-\cos sx}{s^2}$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$~~= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{s^2}\right] = -\sqrt{\frac{2}{\pi}} \times \frac{1}{s^2}~~$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\left[\cancel{(1-1)}\,0 - \frac{\cos s(1)}{s^2}\right] - \left[0 - \frac{1}{s^2}\right]\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 (s/2)}{s^2}\right]$$

$$F^{-1}[F(s)] := f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$1-|x| = 1-x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}} \times 2\,\frac{\sin^2 s/2}{s^2}\,ds$$

$$1-x = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\frac{s}{2}}{s^2}\,e^{-isx}\,ds$$

even

$$1-x = \frac{2}{\pi}\times 2\int_0^{\infty}\frac{\sin^2\frac{s}{2}\cdot\cos sx}{s^2}\,ds$$

Put $\frac{s}{2} = t$ $\quad ds = 2dt$

$x = 0 \quad s = 2t$

$$1-0 = \frac{\cancel{4}2}{\pi}\int_0^{\infty}\frac{(\sin t)^2}{\cancel{4}t^2}\,\cancel{2}dt$$

$$1-0 = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

$$\frac{\pi}{2} = \int_0^{\infty}\left(\frac{\sin t}{2}\right)^2 dt$$

Parseval's Identity

$$\int_{-\infty}^{\infty}(f(x))^2\,dx = \int_{-\infty}^{\infty}[F(s)]^2\,ds$$

even

$$\int_{-1}^{1}(1-|x|)^2\,dx = \int_{-\infty}^{\infty}\left[\left(\sqrt{\frac{2}{\pi}}\right)^2 \times 2\,\frac{\sin^2 s/2}{s^2}\right]^2 ds$$

$$2\int_0^1(1-x)^2\,dx = \int_{-\infty}^{\infty}\frac{8}{\pi}\left[\frac{\sin^4 s/2}{s^4}\right]ds$$

Put $\frac{s}{2} = t$ $\quad s = 2t$

$ds = 2dt$

$$\cancel{2}\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{8}{\pi}\times\cancel{2}\int_0^\infty \left[\left(\frac{\sin t}{2t}\right)^4 2\,dt\right.$$

$$\frac{\cancel{-}1}{\cancel{-}3} = \frac{8}{\pi}\int_0^\infty \cancel{4}\,\frac{\cancel{2}^{3}}{\cancel{t}^4}\,\frac{1}{2^3}\left(\frac{\sin t}{t}\right)^4 dt$$

$$\frac{\pi}{\cancel{24}\,3} = \frac{1}{\cancel{8}}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

$$\frac{\pi}{3} = \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt \;//$$