Harmonic Analysis

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

1. Find the Fourier series expansion defined in $(0, T)$ by means of the table of values given below. Find the series up to the second harmonic.

t-Sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A -amp	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution:

$N = 6$

$T = 2\pi = 360^\circ$

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \dots \rightarrow \quad (1)$$

x	$y = f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.98	1.98	1.98	0	0
$\frac{T}{6}$	1.3	0.65	-0.65	1.126	1.126
$\frac{T}{3}$	1.05	-0.525	-0.525	0.909	-0.909
$\frac{T}{2}$	1.3	-1.3	1.3	0	0
$\frac{2T}{3}$	-0.88	0.44	0.44	0.762	-0.762
$\frac{5T}{6}$	-0.25	-0.125	0.125	0.217	0.217
Σ	4.5	1.12	2.67	3.014	-0.328

$$a_0 = 2*\left(\frac{\Sigma y}{N}\right) = 2*\left(\frac{4.5}{6}\right) = 1.5$$

$$a_1 = 2*\left(\frac{\Sigma y \cos x}{N}\right) = 2*\left(\frac{1.12}{6}\right) = 0.373$$

$$a_2 = 2*\left(\frac{\Sigma y \cos 2x}{N}\right) = 2*\left(\frac{2.67}{6}\right) = 0.89$$

$$b_1 = 2*\left(\frac{\Sigma y \sin x}{N}\right) = 2*\left(\frac{3.014}{6}\right) = 1.005$$

$$b_2 = 2*\left(\frac{\Sigma y \sin 2x}{N}\right) = 2*\left(\frac{-0.328}{6}\right) = -0.109$$

Substituting all these values in (1), we get

$$f(x) = 0.75 + 0.373\cos x + 0.89\cos 2x + 1.005\sin x - 0.109\sin 2x$$

2. Find the Fourier series as far as the second harmonic to represent the function given in the following data.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Solution:

$N = 6$

$2l = 6 \Rightarrow l = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$= \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + a_2 \cos\left(\frac{2\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) + b_2 \sin\left(\frac{2\pi x}{3}\right) + \ldots \rightarrow (1)$$

HARMONIC ANALYSIS

x	$y=f(x)$	$y\cos\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{2\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$	$y\sin\left(\frac{2\pi x}{3}\right)$
0	0	9	9	0	0
1	18	9	-9	15.7	15.6
2	24	-12	-24	20.9	0
3	28	-28	28	0	0
4	26	-13	-13	-22.6	22.6
5	20	10	-10	-17.4	-17.4
Σ	125	-25	-19	-3.4	20.8

$$a_0 = 2 * \left(\frac{\Sigma y}{N}\right) = 2 * \left(\frac{125}{6}\right) = 41.66$$

$$a_1 = 2 * \left(\frac{\Sigma y \cos\left(\frac{\pi x}{3}\right)}{N}\right) = 2 * \left(\frac{-25}{6}\right) = -8.33$$

$$a_2 = 2 * \left(\frac{\Sigma y \cos\left(\frac{2\pi x}{3}\right)}{N}\right) = 2 * \left(\frac{-19}{6}\right) = -6.33$$

$$b_1 = 2 * \left(\frac{\Sigma y \sin\left(\frac{\pi x}{3}\right)}{N}\right) = 2 * \left(\frac{-3.4}{6}\right) = -1.13$$

$$b_2 = 2 * \left(\frac{\Sigma y \sin\left(\frac{2\pi x}{3}\right)}{N}\right) = 2 * \left(\frac{20.8}{6}\right) = 0.009$$

Substituting all these values in (1), we get

$$f(x) = 20.83 - 8.33\cos\left(\frac{\pi x}{3}\right) - 6.33\cos\left(\frac{2\pi x}{3}\right) - 1.13\sin\left(\frac{\pi x}{3}\right) + 0.009\sin\left(\frac{2\pi x}{3}\right)$$

3. Find the Fourier Series up to second Harmonic level for the following data:

x	0	π /3	2π/3	π	4π/3	5π/3	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution:

$N = 6$

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \dots \rightarrow \quad (1)$$

x	$y = f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.0	1	1	0	0
$\frac{\pi}{3}$	1.4	0.7	-0.7	1.2124	1.2124
$\frac{2\pi}{3}$	1.9	-0.95	-0.95	1.6454	-1.6454
π	1.7	-1.7	1.7	0	0
$\frac{4\pi}{3}$	1.5	-0.75	-0.75	-1.299	1.299
$\frac{5\pi}{3}$	1.2	0.6	-0.6	-1.0392	-1.0392
Σ	8.7	-1.1	-0.3	0.5196	-0.1732

$$a_0 = 2*\left(\frac{\Sigma y}{N}\right) = 2*\left(\frac{8.7}{6}\right) = 2.90$$

$$a_1 = 2*\left(\frac{\Sigma y \cos x}{N}\right) = 2*\left(\frac{-1.1}{6}\right) = -0.37$$

$$a_2 = 2*\left(\frac{\Sigma y \cos 2x}{N}\right) = 2*\left(\frac{-0.3}{6}\right) = -0.1$$

$$b_1 = 2*\left(\frac{\Sigma y \sin x}{N}\right) = 2*\left(\frac{0.5196}{6}\right) = 0.17$$

$$b_2 = 2*\left(\frac{\Sigma y \sin 2x}{N}\right) = 2*\left(\frac{-0.1732}{6}\right) = -0.06$$

Substituting all these values in (1), we get

$$f(x) = 1.45 - 0.37\cos x - 0.1\cos 2x + 0.17\sin x - 0.06\sin 2x$$