If the transform of $f(x)$ is $f(s)$ then the function $f(x)$ is called self-reciprocal

14) Show that the function $e^{-x^2/2}$ is self-reciprocal under fourier transform.

soln: Gn: $f(x) = e^{-x^2/2}$

WKT $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{x^2}{2} + isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx + (is)^2 - (is)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 - (is)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2 + \left(\frac{is}{\sqrt{2}}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{\sqrt{2}}\right)^2} \int_{-\infty}^{\infty} e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2 s^2}{2}} \int_{-\infty}^{\infty} e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2} dx.$$

put $t = \frac{x-is}{\sqrt{2}}$. $\Rightarrow dt = \frac{dx}{\sqrt{2}}$.

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \sqrt{2} \int_{-\infty}^{\infty} e^{-t^2} dt.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \sqrt{2}.\sqrt{\pi}$$

$$= e^{-s^2/2}.$$

$\therefore$ The Fourier transform of $e^{-x^2/2}$ (i.e.) $F[e^{-x^2/2}]$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self-reciprocal under Fourier transform.

13> Find the Fourier transform of $f(x) = e^{-a^2x^2}, a>0$. Show that $e^{-x^2/2}$ is self reciprocal under Fourier transformation

Soln:

WKT $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[a^2x^2-isx]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2[x^2-\frac{is}{a^2}x]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2[x^2-\frac{is}{a^2}x+(\frac{is}{2a^2})^2-(\frac{is}{2a^2})^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2\{[x-\frac{is}{2a^2}]^2-(\frac{is}{2a^2})^2\}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\{[ax-\frac{is}{2a}]^2-\frac{i^2s^2a^2}{2^2(a^2)^2}\}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[ax-\frac{is}{2a}]^2-\frac{s^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[ax-\frac{is}{2a}]^2} . e^{-s^2/4a^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-[ax-\frac{is}{2a}]^2} dx$$

put $t = ax - \frac{is}{2a} \Rightarrow dt = a\,dx$.

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} . e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} dt.$$

$$\boxed{\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}dx &= \sqrt{\pi} \\ \int_{0}^{\infty} e^{-x^2}dx &= \frac{\sqrt{\pi}}{2} \end{aligned}}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-s^2/4a^2} \sqrt{\pi}$$

$$F[e^{-a^2x^2}] = e^{-s^2/4a^2} . \frac{1}{a\sqrt{2}}.$$

put $a^2 = 1/2 \Rightarrow a = 1/\sqrt{2}$.

$$F[e^{-x^2/2}] = e^{-\frac{s^2}{4 \cdot \frac{1}{2}}} \frac{1}{1/\sqrt{2} . \sqrt{2}} = e^{-s^2/2} \Rightarrow F[e^{-x^2/2}] = e^{-s^2/2},$$ which is self-reciprocal under FT.