FOURIER SINE & COSINE TRANSFORM WITH PARSEVAL'S IDENTITY:

SINE TRANSFORM:

The Fourier sine transform of a function $f(x), 0<x<\infty$ is defined as $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

The Inverse Fourier sine transform of $F_s(s)$ is defined as $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$.

Parseval's Identity: If $F_s(s)$ is the Fourier transform of $f(x)$ then $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$.

NOTE: $F_s(s)$ and $F^{-1}[F_s(s)]$ is called Fourier sine transform pair.

COSINE TRANSFORM:

The Fourier cosine transform of a function $f(x), 0<x<\infty$ is defined as $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$.

The Inverse Fourier cosine transform of $F_c(s)$ is defined as $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$.

If $F_c(s)$ is the Fourier transform of $f(x)$ then Parseval's Identity is $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

NOTE: $F_c(s)$ and $F^{-1}[F_c(s)]$ is called Fourier cosine transform pair

1) Find the Fourier sine transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$$

soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s}\right]$$

2) Find the Fourier sine transform of $1/x$.

Soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} dx$$

putting $\theta = sx \Rightarrow d\theta = s\,dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta/s} d\theta/s$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} d\theta . \quad \left[\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

$$= \sqrt{\pi/2}$$

3) Find the Fourier cosine transform of $2e^{-3x} + 3e^{-2x}$.

Soln: WKT $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[2\left[\frac{3}{s^2+9}\right] + 3\left[\frac{2}{s^2+4}\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{6}{s^2+9} + \frac{6}{s^2+4}\right]$$

4) Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & \text{if } 0<x<a \\ 0, & \text{if } x \geq a \end{cases}$

Soln: WKT $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos x . \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2}\left[\cos(s+1)x + \cos(s-1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a \left[\cos(s+1)x + \cos(s-1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]$$

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5) Find the Fourier sine & cosine transform of e^{-ax} and deduce that inverse Fourier transform & parseval's identity

soln:

Sine Transform:

$$\text{WKT } F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Inverse Transform:

$$\text{WKT } f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

Parseval's Identity:

$$\text{WKT } \int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]\right]^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

Cosine Transform:

$$\text{WKT } F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Inverse Transform :

WKT $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{a}{s^2+a^2} \cdot \cos sx \, ds$$

put $x = 0$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{a}{s^2+a^2} \cos s(0) \, ds$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{a}{s^2+a^2} \, ds .$$

Parseval's Identity :

WKT $\int_0^{\infty} [f(x)]^2 dx = \int_0^{\infty} [F_c(s)]^2 ds$

$$\int_0^{\infty} [e^{-ax}]^2 dx = \int_0^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right]^2 ds$$

$$\int_0^{\infty} e^{-2ax} dx = \frac{2}{\pi} \int_0^{\infty} \left[\frac{a}{s^2+a^2}\right]^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty} : \frac{1}{2a} = \frac{2}{\pi} \int_0^{\infty} \left[\frac{a}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^{\infty} \left[\frac{a}{s^2+a^2}\right]^2 ds .$$

PROPERTIES OF FOURIER TRANSFORMS :

1) LINEAR PROPERTY :

Show that the operator 'F' is linear.

(a) $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$

Now $F[af(x)+bg(x)]$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x)e^{isx}] + [bg(x)]e^{isx} dx.$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= aF[f(x)] + bF[g(x)]$$, F is linear.

$|||^{ly}$ $F_s[af(x)+bg(x)] = aF_s[f(x)] + bF_s[g(x)]$

$F_c[af(x)+bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

2) SHIFTING PROPERTY :

(i) $F[f(x-a)] = e^{isa} F(s)$

(ii) $F[e^{iax} f(x)] = F(s+a)$

(i) $F[f(x-a)] = e^{isa} F(s)$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Now $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$

put $x-a = p \Rightarrow dx = dp.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) e^{is(a+p)} dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) e^{isa} . e^{isp} dp$$

$$= e^{isa} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) e^{isp} dp$$

$$= e^{isa} F(s)$$

(ii) $F[e^{iax} f(x)] = F(s+a)$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Now $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$

$= F(s+a)$

3> CHANGE OF SCALE PROPERTY:

$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right),\ a > 0$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$

put $t = ax \Rightarrow dt = a\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist/a} \frac{dt}{a}$

$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist/a} dt$

$= \frac{1}{a} F\left(\frac{s}{a}\right)$

III^{ly} $F_c[f(ax)] = \frac{1}{a} F_c\left[\frac{s}{a}\right]$

$F_s[f(ax)] = \frac{1}{a} F_s\left[\frac{s}{a}\right]$

4) $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Now $\frac{d}{ds} F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix) dx$

$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{isx} dx.$

$= i F[x f(x)]$

$\Rightarrow \frac{d}{ds} F = i F[x f(x)]$

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$$\frac{d^2F}{ds^2} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{isx} (ix) dx$$

$$= \frac{i^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 f(x) e^{isx} dx.$$

$$= \frac{i^2}{\sqrt{2\pi}} \int_{-\infty} [x^2 f(x)] e^{isx} dx$$

$$= i^2 F[x^2 f(x)]$$

$$|||^{ly} \quad \frac{d^nF}{ds^n} = i^n F[x^n f(x)]$$

$$\Rightarrow F[x^n f(x)] = (-i)^n \frac{d^nF}{ds^n}$$

$$\left(\frac{1}{i}\right)^n = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$$

5) **MODULATION (PROPERTY) THEOREM :**

(i) $$F[f(x) \cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

WKT $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Now $$F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} \left[f(x) e^{ix(s+a)} + f(x) e^{ix(s-a)}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

(ii) $$F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

WKT $$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$$

Now $$F_s[f(x) \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \frac{1}{2} [\sin(s+a)x + \sin(s-a)] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(s+a)x \, dx + \sqrt{\frac{2}{\pi}} \int_0 f(x) \sin(s-a)x \, dx\right]$$

$$= \frac{1}{\cancel{\sqrt{2}}2}[F_s(s+a) + F_s(s-a)]$$

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IIIly (iii) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a+s) + F_c(a-s)]$

(iv) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(v) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

6) (i) $F_c[xf(x)] = \frac{d}{ds} F_s[f(x)]$

(ii) $F_s[xf(x)] = -\frac{d}{ds}[F_c[f(x)]$

(i) WKT $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx \, dx$

$$\frac{d}{ds} F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx \cdot x\, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x)\cos sx \, dx$$

$$= F_c[xf(x)]$$

(ii) WKT $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx \, dx$

$$\frac{d}{ds} F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)(-\sin sx).x\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x)\sin sx \, dx .$$

$$= -F_s[xf(x)]$$

7) (i) $F[f'(x)] = -isF(s)$.

(ii) $F[f^n(x)] = (-i)^n s^n F(s)$.

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} dx$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f'(x) e^{isx} dx .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{isx} d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x)\right]_{-\infty}^\infty - \int_{-\infty}^\infty f(x) e^{isx}(is)\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 0 - is \int_{-\infty}^\infty f(x) e^{isx} dx \right\}$$

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} dx$$

$$= -is\, F(s) .$$

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|||ly $F[f^n(x)] = (-is)^n F(s)$

8) $F\left[\int_a^x f(x)\,dx\right] = \dfrac{F(s)}{(-is)}$

9)(i) $F[\overline{f(x)}] = \overline{F(-s)}$

(ii) $F[\overline{f(-x)}] = \overline{F(s)}$.