Unit - II

Fourier Transform

The Fourier Transform of a function is $F(S) = F(f(x))$

$$F(S) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(S)\, e^{-isx}\, dx$$

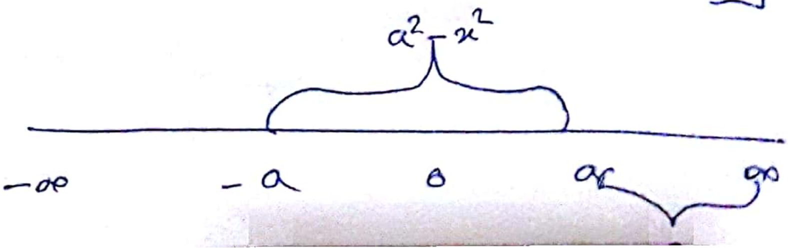
(1) P.T the Fourier transform $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

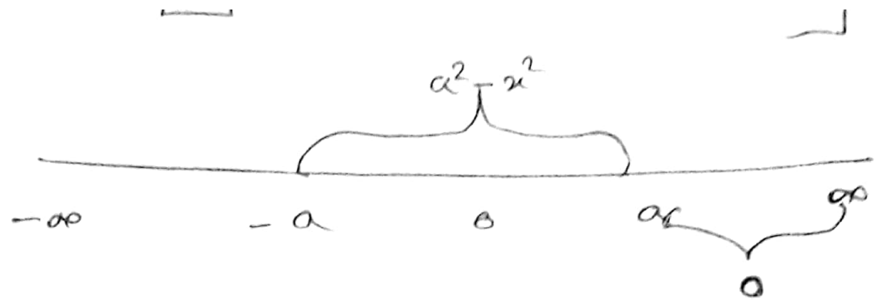
is $2\sqrt{\frac{2}{\pi}} \left\{ \frac{\sin as - as\cos as}{s^2} \right\}$ and deduce $\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$

and also using parseval's Identity. P.T $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{15}$

Soln

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad \left[\because e^{isx} = \cos sx + i \sin sx \right]$$





$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} (a^2-x^2)(\cos sx + i \sin sx)\, dx\right.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} (a^2-x^2) \cos sx\, dx + i \underbrace{\int_{-a}^{a} \cancel{(a^2-x^2) \sin sx\, dx}}_{0 \text{ odd fun}}\right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} a^2-x^2 \cos sx\, dx$$

$u = a^2 - x^2 \qquad dv = \cos sx \, dx$

$u' = -2x \qquad v_1 = \frac{\sin sx}{s}$

$u'' = -2 \qquad v_2 = \frac{-\cos sx}{s^2}$

$u''' = 0 \qquad v_3 = -\frac{\sin sx}{s^3}$

$$F(s) = \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[-2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3} \right]$$

$$= \frac{2 \times 2}{\sqrt{2\pi}} \left[\frac{\sin sa}{s^3} - \frac{a\cos sa}{s^2} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right]$$

$$= \frac{2 \times 2}{\sqrt{2\pi}} \left[\frac{\sin as - as\cos sa}{s^3} \right]$$

$$= \frac{2 \times 2}{\sqrt{2}\sqrt{\pi}} \left[\frac{\sin as - as\cos sa}{s^3} \right]$$

$$= \frac{2 \times \sqrt{2}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{\pi}} \left[\frac{\sin as - as\cos sa}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as\cos sa}{s^3} \right]$$

Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} \, ds$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as-as\cos sa}{s^3}\right]e^{-isx}\,ds$$

$$\left[\because e^{-isx}=\cos sx-i\sin sx\right]$$

$$=\frac{1}{\sqrt{2}\sqrt{\pi}}\frac{2\sqrt{2}}{\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as-as\cos sa}{s^3}\right)\cos sx\,ds-i\int_{-\infty}^{\infty}\left(\frac{\sin as-as\cos sa}{s^3}\right)\sin sx\,ds\right]$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as-as\cos sa}{s^3}\right)\cos sx\,ds$$

$$f(x)=\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin as-as\cos sa}{s^3}\right)\cos sx\,ds$$

Sub $x=0$, $a=1$

$$f(0)=\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\cos 0\,ds$$

$$1=\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s-s\cos s}{s^3}\,ds$$

$$\left[\begin{aligned}&f(x)=a^2-x^2\\&f(0)=a^2\\&f(0)=1\end{aligned}\right]$$

Replace 's' by 't'

$$\frac{\pi}{4}=\int_{0}^{\infty}\left(\frac{\sin t-t\cos t}{t^3}\right)dt$$

Parsaval's Identity

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds=\int_{-\infty}^{\infty}(f(x))^2\,dx$$

$$\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as-as\cos sa}{s^3}\right]\right]^2 ds=\int_{-a}^{a}(a^2-x^2)^2\,dx$$

sub $a=1$

$$\frac{4\times 2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = 2\int_0^1 (1^2 - x^2)^2\, dx$$

sub $s = -s$

$$\frac{\sin(-s) - (-s)\cos(-s)}{(-s)^3}$$

$$\frac{-\sin s + s\cos s}{-(s)^3} = \frac{\cancel{+}(\sin s - s\cos s)}{\cancel{+}(s)^3}$$

$$f(-s) = f(s)$$

$\therefore$ even

$$\frac{8\times 2}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = 2\int_0^1 \left(1 - 2x^2(1) + (x^2)^2\right) dx$$

$$= 2\int_0^1 (1 - 2x^2 + x^4)\, dx$$

$$= 2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= 2\left[1 - \frac{2(1)}{3} + \frac{1}{5}\right] = 2\left[\frac{15-10+3}{15}\right]$$

$$= 2\left[\frac{8}{15}\right] = \frac{16}{15}$$

$$\therefore \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{16}{15}$$

$$\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\cancel{16}}{15} \times \frac{\pi}{\cancel{16}} = \frac{\pi}{15}$$