Half Range Expansions.

In many Engineering problems it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or in the range $(0, l)$ in a Fourier series of period $2l$.

The half range cosine series in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The half-range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

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Problems on Fourier cosine series

1. Obtain the Fourier expansion of $x\sin x$ as a cosine series in $(0,\pi)$ and hence deduce the value of

$$1+\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}-\cdots$$

Soln: The Fourier cosine series of $f(x)$ in $(0,\pi)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx.$$

To find a_0:-

$$a_0=\frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$u=x$ $\quad$ $v=\sin x$

$u'=1$ $\quad$ $v_1=-\cos x$

$u''=0$ $\quad$ $v_2=-\sin x$

$$=\frac{2}{\pi}\Big[-x\cos x-(1)(-\sin x)\Big]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\pi\cos\pi+0+0-0\right]$$

$$=\frac{2}{\cancel{\pi}}(-\cancel{\pi}\cos\pi)$$

$$=-2(-1)$$

$$\boxed{a_0=2}$$

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To find a_n:-

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underline{x\sin x\cos nx}\,dx.$$

$$\left[\sin x\cos nx = \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,\frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]dx.$$

$$= \frac{\cancel{2}}{\pi}\frac{1}{\cancel{2}}\left[\int_0^{\pi} x\sin(1+n)x\,dx + \int_0^{\pi} x\sin(1-n)x\,dx\right]$$

$u = x$	$v = \sin(1+n)x$	$u = x$	$v = \sin(1-n)x$
$u' = 1$	$v_1 = -\frac{\cos(1+n)x}{1+n}$	$u' = 1$	$v_1 = -\frac{\cos(1-n)x}{1-n}$
$u'' = 0$	$v_2 = -\frac{\sin(1+n)x}{(1+n)^2}$	$u'' = 0$	$v_2 = -\frac{\sin(1-n)x}{(1-n)^2}$

$$= \frac{1}{\pi}\left[\left\{-x\frac{\cos(1+n)x}{1+n} + \frac{\sin(1+n)x}{(1+n)^2}\right\}_0^{\pi}\right.$$

$$\left. + \left\{-x\frac{\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{(1-n)^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(1+n)\pi}{1+n} - \frac{\pi\cos(1-n)\pi}{1-n}\right]$$

$$= \frac{1}{\cancel{\pi}}(-\cancel{\pi})\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right]$$

$$= (-1)\left[\frac{\cos\pi\cos n\pi}{1+n} + \frac{\cos\pi\cos n\pi}{1-n}\right]$$

$$= (-1)\cos\pi\cos n\pi\left[\frac{1}{1+n} + \frac{1}{1-n}\right].$$

$$= (-1)(-1)(-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= (-1)^n\left[\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right]$$

$$= \frac{(-1)^n 2}{1-n^2}.$$

$$\boxed{a_n = \frac{2(-1)^n}{1-n^2}}$$ provided $n \neq 1$

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when $n=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$[\sin 2x = 2\sin x \cos x]$$

$u = x$ $\qquad$ $v = \sin 2x$

$u' = 1$ $\qquad$ $v_1 = -\frac{\cos 2x}{2}$

$u'' = 0$ $\qquad$ $v_2 = -\frac{\sin 2x}{4}$

$$= \frac{1}{\pi}\left[-x\frac{\cos 2x}{2} - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos 2\pi}{2} + 0\right]$$

$$\boxed{a_1 = -1/2}$$

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∴ The Fourier cosine series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx .$$

$$= \frac{2}{2} + \left(-\frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx .$$

$$= 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2} \cos nx .$$

$$x\sin x = 1 - \frac{\cos x}{2} - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx .$$

Deduction:-

put $x = \pi/2$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos\frac{n\pi}{2}}{(n-1)(n+1)}$$

$$\frac{\pi}{2} = 1 - 0 - 2\sum_{n=2,4,..}^{\infty} \frac{(-1)^n \cos\frac{n\pi}{2}}{(n-1)(n+1)}$$

[∵ $\cos\frac{n\pi}{2} = 0$ when n is odd]

$$\frac{\pi}{2} = 1 - 2\left[\frac{(-1)^2\cos\pi}{(2-1)(2+1)} + \frac{(-1)^4\cos 2\pi}{(4-1)(4+1)} \cdots\right]$$

$$= 1 - 2\left[\frac{(-1)}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} \cdots\right]$$

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$$\frac{\pi}{2} = 1 + 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$

$$\frac{\pi - 2}{2} = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\therefore \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots = \frac{\pi - 2}{4}.$$

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(2) Find the sine series of $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:- $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx.$$

$u = x\pi - x^2$ $\qquad$ $v = \sin nx$

$u' = \pi - 2x$ $\qquad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = -2$ $\qquad$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\qquad$ $v_3 = \frac{\cos nx}{n^3}$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 0 - \frac{2\cos n\pi}{n^3} + 0 - 0 + \frac{2\cos 0}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

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$$= \frac{2}{\pi}\left(\frac{2}{n^3}\right)\left[-(-1)^n+1\right]$$

$$b_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$$

$\therefore$ The Sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right)\sin nx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^3}\left(1-(-1)^n\right)\sin nx$$

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