FOURIER TRANSFORM

Fourier transform of $f(x)$ is

$$F(s) = F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx .$$

$(e^{isx} = \cos sx + i \sin sx)$

Inversion formula for Fourier transform $F(s)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \, e^{-isx} \, ds .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x)) \, e^{-isx} \, ds .$$

$(e^{-isx} = \cos sx - i \sin sx)$

Parseval's Identity :

If $F(s)$ is the fourier transform of $f(x)$. Then

$$\int_{-\infty}^{\infty} [f(x)]^2 \, dx = \int_{-\infty}^{\infty} [F(s)]^2 \, ds$$

1) Find the Fourier transform of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

Soln: The gn. function can be written as.

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & a < x < \infty \quad -\infty < x < \infty \end{cases}$$

WKT $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0\, dx + \int_{-a}^{a} x e^{isx} dx + \int_{a}^{\infty} 0\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{isx}}{is} - \frac{e^{isx}}{(is)^2} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a e^{isa}}{is} - \frac{e^{isa}}{(is)^2} - \left[-a \frac{e^{-isa}}{is} - \frac{e^{-isa}}{(is)^2} \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a e^{ias}}{is} - \frac{e^{isa}}{(is)^2} + \frac{a e^{-isa}}{is} + \frac{e^{-isa}}{(is)^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{is} \left[e^{ias} + e^{-ias} \right] - \frac{1}{(is)^2} \left[e^{ias} - e^{-ias} \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{is}\, 2\cos as - \frac{1}{(is)^2}\, 2i \sin as \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{is}\, 2\cos as + \frac{1}{s^2}\, 2i \sin as \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{is} \cos as + \frac{i}{s^2} \sin as \right]$$

$$= \sqrt{\frac{2}{\pi}}\, i \left[\frac{a}{i^2 s} \cos as + \frac{\sin as}{s^2} \right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s^2} - \frac{a\cos as}{s}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^2}\right].$$