PARSEVAL'S IDENTITY :

Let $f(x)$ be a periodic function with period 2π defined in the interval $(c, c+2\pi)$ then

$$\frac{1}{2\pi}\int_c^{c+2\pi} [f(x)]^2\,dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(b_n)^2 .$$

Where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$.

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx .$$

1) Find the Fourier series of the function $f(x) = x^2$ in $-\pi < x < \pi$ and show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$.

Soln:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

NKT $a_0 = \frac{2\pi^2}{3}$; $a_n = \frac{4(-1)^n}{n^2}$; $b_n = 0$.

Deduction:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \left(\frac{\cancel{2}\pi^2}{3\times\cancel{2}}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 + 0 .$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{\pi^4}{9} + \frac{1}{2}\times 16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4}.$$

$$\frac{1}{\cancel{2}\pi} \cdot \cancel{2}\int_0^{\pi} x^4\, dx = \frac{\pi^4}{9} + 8\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} - \frac{\pi^4}{9} = 8\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\left[\frac{\pi^4}{5} - \frac{\pi^4}{9}\right]\frac{1}{8} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\frac{\cancel{4}\pi^4}{45} \times \frac{1}{\cancel{8}_2} : \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

ROOT MEAN SQUARE (RMS)

Let $f(x)$ be a function defined in an interval (a, b) then $\sqrt{\frac{\int_a^b (f(x))^2 dx}{(b-a)}}$ is called the root mean square or effective value of $f(x)$ and is denoted by $\bar{y}$.

i.e., $\bar{y}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx$.

1) Find the RMS value of $y = x^2$ in $(-\pi, \pi)$

Soln: $\bar{y}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx$

$= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}(x^2)^2 dx$

$= \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi}$

$= \frac{1}{10\pi}\left[\pi^5 - (-\pi)^5\right]$

$= \frac{2\pi^5}{10\pi} = \frac{\pi^4}{5}.$

$\bar{y} = \frac{\pi^2}{\sqrt{5}}.$

2) Find the RMS value of $y = x$ in $0 < x < l$.

Soln: $\bar{y}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx$

$= \frac{1}{l}\int_0^l x^2 dx.$

$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$

$\bar{y} = \frac{l}{\sqrt{3}}$