Half Range series :-

Sine series :

The half-range sine series of $f(x)$ defined in the interval $0<x<\pi$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx .$$

1) Express $f(x) = x(\pi - x)$, $0<x<\pi$ as a Fourier series of periodicity 'π' containing (i) sine terms

Soln: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2\cos n\pi}{n^3} - \left(0 - \frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{\pi n^3}\left[-(-1)^n + 1\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left[1 - (-1)^n\right] \sin nx$$

Find the Half range series for $f(x) = \begin{cases} \frac{\pi x}{4}, & 0 < x < \pi/2 \\ \frac{\pi}{4}(\pi - x), & \pi/2 < x < \pi \end{cases}$

(i) Half-range cosine series:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi x}{4}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{\pi}{4}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\cdot\frac{\pi^2}{8} + \frac{\pi}{4}\left(\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{32} + \frac{\pi^3}{8} - \frac{3\pi^3}{32}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{8} - \frac{2\pi^3}{3\cancel{2}_{16}}\right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\pi^{\cancel{3}\,2}}{\cancel{16}\,8} = \frac{\pi^2}{8}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{\pi}{4}x\cos nx\,dx + \int_{\pi/2}^{\pi}\frac{\pi}{4}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{4}\left[x\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \right.$$

$$\left.\frac{\pi}{4}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\times\frac{\cancel{\pi}}{\cancel{4}\,2}\left\{\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{\cos 0}{n^2}\right] + \right.$$

$$\left.\left[0 - \frac{\cos n\pi}{n^2} - \left(\frac{\pi}{2}\frac{\sin n\pi/2}{n} - \frac{\cos n\pi/2}{n^2}\right)\right]\right\}$$

$$= \frac{1}{2}\left[\cancel{\frac{\pi}{2}\frac{\sin n\pi/2}{n}} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2}\frac{\sin n\pi/2}{n}} + \frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{1}{2}\left[\frac{2\cos n\pi/2 - 1 - (-1)^n}{n^2}\right]$$

$$\therefore f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{2\cos n\pi/2 - 1 - (-1)^n}{n^2}\right]\cos nx.$$

SINE SERIES:

The half range sine series in the interval $(0, l)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$.

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$.

Find the Fourier sine series expansion for $f(x) = x$ in $(0, l)$

Soln: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$.

Now $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}\right) x \, dx$

$$= \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi}{l}\right) x \, dx.$$

$$= \frac{2}{l} \left[x \left(-\frac{\cos (n\pi/l) x}{(n\pi/l)} \right) - (1) \left(-\frac{\sin (n\pi/l) x}{(n\pi/l)^2} \right) \right]_0^l$$

$$= \frac{2}{l} \left[-l \cdot \frac{\cos n\pi}{n\pi/l} \right]$$

$$= -\frac{2l}{n\pi} (-1)^n = \frac{2l}{n\pi} (-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi}{l}\right) x.$$