Parseval's Identity

(*) If the Fourier Series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(*) For $[0, l)$

Cosine Series $\quad \frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

Sine Series $\quad \frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$

Problem:

1) Find the Fourier Series of $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

(iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

Solution:

Given $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is even and $\therefore b_n = 0$

a_0 and a_n exists.

Step 1:

The Fourier Series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Step 2:

$$a_0 = \frac{1}{l}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{l}\int_{0}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2\pi^3}{\pi(3)}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

Step 3: To find a_n $\quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos nx \, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$u = x^2$ $\quad dv = \cos nx \, dx$

$u' = 2x$ $\quad v = \frac{1}{n}\sin nx$

$u'' = 2$ $\quad v_1 = \frac{-1}{n^2}\cos nx$

$u''' = 0$ $\quad v_2 = \frac{-1}{n^3}\sin nx$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{1}{n}\sin nx\right) - 2x\left(\frac{-1}{n^2}\cos nx\right) + 2\left(\frac{-1}{n^3}\sin nx\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\pi\left(\frac{-1}{n^2}\right)(-1)^n\right]$$

$$= \frac{4\pi}{\pi n^2}(-1)^n = \frac{4}{n^2}(-1)^n$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Step 4: Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2} \quad \text{--- (1)}$$

Step 5: Deduction.

put $x = 0$ in (1)

$$f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos(0)}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{+(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{——— (2)}$$

put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{(-1)(-1)}{1^2} + \frac{(-1)^2(-1)^2}{2^2} + \frac{(-1)^3(-1)^3}{3^2} + \cdots\right]$$

(or)

$$\frac{3\pi^2 - \pi^2}{3} = 4\left[\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}\right]$$

$$\frac{2\pi^2}{3} = 4\left[\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{(-1)^2}{1^2} + \frac{(-1)^4}{2^2} + \frac{(-1)^6}{3^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{——— (3)}$$

~~put~~ Now Adding (2) & (3) we get

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right] + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{3\pi^2}{12} = \left[\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right]$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \quad //$$

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Using Parseval's Identity:

In the interval $(-\pi, \pi)$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2\,dx = \frac{(2\pi^2/3)^2}{2} + \sum_{n=1}^{\infty}\left(\left(\frac{4}{n^2}(-1)^n\right)^2 + 0\right)$$

$$\frac{2}{\pi}\int_{0}^{\pi}[x^2]^2\,dx = \frac{4\pi^4/3^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4^2}{(n^2)^2}(-1)^{2n}\right)$$

$$\frac{2}{\pi}\int_{0}^{\pi}x^4\,dx = \frac{4\pi^4}{6\times 3} + 16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4}$$

$$\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{4\pi^4}{18} + 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{5} - 0\right] = \frac{2\pi^4}{9} + 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{18\pi^4 - 10\pi^4}{45\times 16} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\frac{8\pi^4}{45\times 16} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \;//$$

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2) Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$ and hence prove that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}.$$

Solution:

Step 1: The half range cosine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] = \frac{\pi^2}{\pi} = \pi. \quad \boxed{a_0 = \pi}$$

Step 3: To find a_n

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi} \left[x \left(\frac{1}{n} \sin nx \right) - \left(- \frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \left\{ \frac{\cos 0}{n^2} \right\} \right]$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{2}{n^2 \pi} \left[(-1)^n - 1 \right]$$

$$a_n = \begin{cases} 0, & \text{when '}n\text{' is even} \\ \dfrac{-4}{n^2 \pi}, & \text{when '}n\text{' is odd} \end{cases}$$

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Step 4: The required Fourier Cosine Series

Substituting ② & ③ in ① we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \left(\frac{-4}{n^2\pi}\right) \cos nx$$

The Parseval's identity for Fourier Cosine Series in $(0, \pi)$ is $\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

Here $a_0 = \pi$; $a_n = \frac{-4}{n^2\pi}$, $n \to$ odd

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^2}$$

$$\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2}{\pi}\left(\frac{\pi^3}{3}\right) - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{4\pi^2 - 3\pi^2}{6} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\therefore \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96} \quad //$$

Step 4: The required Fourier Cosine Series

Substituting (2) & (3) in (1) we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \left(\frac{-4}{n^2\pi}\right) \cos nx$$

The Parseval's identity for fourier Cosine Series in $(0,\pi)$ is $\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

Here $a_0 = \pi$; $a_n = \frac{-4}{n^2\pi}$, $n \to$ odd

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^2}$$

$$\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2}{\pi}\left(\frac{\pi^3}{3}\right) - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{4\pi^2 - 3\pi^2}{6} = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\therefore \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96} \quad //$$