3) Determine the Fourier Series for the function

$f(x) = \pi x, \quad 0 \le x \le 1$

$\quad = \pi(2-x), \quad 1 \le x \le 2$

Solution:

Step 1: Fourier Series of $f(x)$

Here $c = 0$; $c + 2l = 2$

$\Rightarrow l = 1$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

Step 2: To find a_0

$$a_0 = \frac{1}{l}\int_0^2 f(x)\,dx = \frac{1}{l}\left\{\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right\}$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \left[\pi \frac{x^2}{2}\right]_0^1 + \left[\pi\left(2x - \frac{x^2}{2}\right)\right]_1^2 dx$$

$$= \pi\left[\frac{1}{2} - 0\right] + \left[\pi\left[\left(2(2) - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]\right]$$

$$= \frac{\pi}{2} + \pi\left(\frac{4}{2} - \frac{3}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\boxed{a_0 = \pi}$$

Step 3: To find a_n

$$a_n = \frac{1}{l}\int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\int_0^1 x\cos n\pi x\,dx + \pi\int_1^2 (2-x)\cos n\pi x\,dx$$

$u = x \qquad dv = \cos n\pi x$

$du = dx \qquad v = \frac{\sin n\pi x}{n\pi}$

$\int u\,dv = uv - \int v\,du$

$= x\frac{\sin n\pi x}{n\pi} - \int \frac{\sin n\pi x}{n\pi}\,dx$

$= \frac{x \sin n\pi x}{n\pi} + \frac{1}{n^2\pi^2}\cos n\pi x$

$u = 2-x \qquad dv = \cos n\pi x$

$du = -dx \qquad v = \frac{\sin n\pi x}{n\pi}$

$\int u\,dv = uv - \int v\,du$

$= (2-x)\frac{\sin n\pi x}{n\pi} - \int \frac{\sin n\pi x}{n\pi}(-dx)$

$= (2-x)\frac{\sin n\pi x}{n\pi} - \frac{1}{n^2\pi^2}\cos n\pi x$

$$a_n = \pi\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$+\pi\left[(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right] + \pi\left[0 - \frac{1}{n^2\pi^2} - 0 + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}\left[(-1)^n - 1\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{-4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

<u>Step 4:</u> To find b_n

$$b_n = \frac{1}{l}\int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\int_0^1 x\sin n\pi x\,dx + \pi\int_1^2 (2-x)\sin n\pi x\,dx$$

$u = x \qquad dv = \sin n\pi x\,dx$

$du = dx \qquad v = \frac{-\cos n\pi x}{n\pi}$

$$\int u\,dv = x\left(-\frac{\cos n\pi x}{n\pi}\right) - \int -\frac{\cos n\pi x}{n\pi}\,dx$$

$$= -\frac{x\cos n\pi x}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x$$

$u = 2-x \qquad dv = \sin n\pi x\,dx$

$du = -dx \qquad v = -\frac{\cos n\pi x}{n\pi}$

$$\int u\,dv = (2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \int -\frac{\cos n\pi x}{n\pi}$$

$$= \frac{(x-2)}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x.$$

$$b_n = \pi\left[\frac{-x\cos n\pi x}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1$$

$$+\pi\left[\frac{(x-2)}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_1^2$$

$$= \pi\left[\frac{-(-1)^n}{n\pi} + 0 + 0 + 0\right] + \pi\left[0 - 0 - \frac{(1-2)}{n\pi}(-1)^n - 0\right]$$

$$= \pi\left[\frac{-(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi}\right] \cancel{+ \frac{2}{n}\left[(-1)^n\right]}$$

$$\boxed{b_n = 0}$$

Step 5: Fourier Series

Substituting a_0, a_n, b_n in step 1

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{4}{n^2\pi} \cos n\pi x$$

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4) Express $f(x) = (\pi - x)^2$ as a Fourier Series of period 2π in the interval $0 < x < 2\pi$. Hence deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$ (HW)

Solution:

Step 1: Fourier series of $f(x)$ in $(0, 2\pi)$

Here $c = 0$; $c + 2l = 2\pi \Rightarrow l = \pi$ | $f(x) = (\pi - x)^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Step 2: To find a_0

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\,dx \qquad = \frac{1}{\pi}\left[2(\pi - x)(-1)\right]_0^{2\pi}$$

$$= \frac{-2}{\pi}\left[(\pi - 2\pi) - \pi\right] = \frac{-2}{\pi}(-2\pi) = \frac{+4\pi}{\pi} = +$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{3}\right]_0^{2\pi}\left[\frac{1}{-1}\right]$$

$$= \frac{-1}{3\pi}\left[(\pi - 2\pi)^3 - (\pi)^3\right]$$

$$= \frac{-1}{3\pi}\left[(-\pi)^3 - (\pi)^3\right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

Step 3: To find a_n

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\cos(nx)\,dx$$

By Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$u = (\pi-x)^2$ | $dv = \cos(nx)\,dx$

$u' = 2(\pi-x)(-1)$ | $v = \left(\frac{1}{n}\right)\sin(nx)$

$u'' = -2(-1)$ | $v_1 = \frac{-1}{n^2}\cos(nx)$

$= 2$ | $v_2 = \left(\frac{-1}{n^3}\right)\sin(nx)$

$$a_n = \frac{1}{\pi}\left[(\pi-x)^2\left(\frac{1}{n}\right)\sin(nx) - 2(\pi-x)(-1)\left(\frac{-1}{n^2}\right)\cos(nx) + 2\left(\frac{-1}{n^3}\right)\sin(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^2}{n}\sin(nx) - \frac{2(\pi-x)}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]$$

$$= \frac{1}{\pi}\left\{\left[0 - \frac{2(\pi-2\pi)}{n^2}(1) - 0\right] - \left[0 - \frac{2\pi}{n^2}(1) - 0\right]\right\}$$

$$= \frac{1}{\pi}\left[+\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{4\pi}{\pi(n^2)} = \frac{4}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

Step 4: To find b_n

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin(nx)\,dx$$

By Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$u = (\pi - x)^2$$
$$u' = 2(\pi - x)(-1) = -2(\pi - x)$$
$$u'' = -2(-1) = 2$$

$$dv = \sin(nx)\,dx$$
$$v = -\cos(nx)\left(\frac{1}{n}\right)$$
$$v_1 = \left(-\frac{1}{n^2}\right)\sin(nx)$$
$$v_2 = \left(\frac{1}{n^3}\right)\cos(nx)$$

$$b_n = \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-1}{n}\right)\cos(nx) + 2(\pi - x)\left(\frac{-1}{n^2}\right)\sin(nx) + 2\left(\frac{1}{n^3}\right)\cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(\pi - x)^2}{n}\cos(nx) - \frac{2(\pi - x)}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{(\pi - 2\pi)^2}{n}(1) - 0 + \frac{2}{n^3}(1)\right] - \left[-\frac{\pi^2}{n} - 0 + \frac{2}{n^3}(1)\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right\} = 0 \qquad \boxed{b_n = 0}$$

Step 5: Fourier Series

Substituting a_0, a_n and b_n in step 1

$$f(x) = \frac{(2\pi^2/3)}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) + \sum_{n=1}^{\infty}(0)\sin(nx)$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$ is the req. fourier series