Harmonic Analysis

Definition: The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Except degree, for all values in Key radian mode

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + a_3 \cos\frac{3\pi x}{l} + \cdots + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots$$

where

$$a_0 = 2 \times \text{mean value of } y = 2\frac{\Sigma y}{N}$$

$$a_n = 2 \times \text{mean value of } y\cos\left(\frac{n\pi x}{l}\right) = 2\frac{\Sigma y\cos\left(\frac{n\pi x}{l}\right)}{N}$$

$$b_n = 2 \times \text{mean value of } y\sin\left(\frac{n\pi x}{l}\right) = 2\frac{\Sigma y\sin\left(\frac{n\pi x}{l}\right)}{N}$$

and N = number of non-repeated values.

Note:

- ★ The term $a_1 \cos x + b_1 \sin x$ is called fundamental or first harmonic
- ★ The term $a_2 \cos 2x + b_2 \sin 2x$ is called second harmonic.
- ★ T denotes period of a given function
- ★ $\underbrace{T = 2\pi = 360}_{\text{all are same}}$ ($\& \ \pi = 180°$)

① The following table gives the variation of a periodic function over a period T.

class

x :	0	T/6	T/3	T/2	2T/3	5T/6	T
$f(x)$:	1.98	1.3	1.05	1.3	−0.88	−0.25	1.98

Find the Fourier Series upto Second harmonic.

Soln:

$N = 6$ $\qquad 2l = 2\pi$

$T = 2\pi = 360°$ $\qquad l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Calculator $\cos\left(\frac{2\pi}{6}\right)\times 1.3 = 0.65$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \dots\dots$$

x	$y = f(x)$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.98	1.98	1.98	0	0
T/6	1.3	0.65	−0.65	1.126	1.126
T/3	1.05	−0.525	−0.525	0.909	−0.909
T/2	1.3	−1.3	1.3	0	0
2T/3	−0.88	0.44	0.44	0.762	−0.762
5T/6	−0.25	−0.125	0.125	0.217	0.217
T	~~1.98~~				
	4.5	1.12	2.67	3.014	−0.328

$$a_0 = 2 \times \left(\frac{\Sigma y}{N}\right) = 1.5$$

$$a_1 = 2 \times \frac{\Sigma y \cos x}{N} = 0.373$$

$$a_2 = 2 \times \frac{\Sigma y \cos 2x}{N} = 0.89$$

$$b_1 = 2 \times \frac{\Sigma y \sin x}{N} = 1.005$$

$$b_2 = 2 \times \frac{\Sigma y \sin 2x}{N} = -0.109$$

$$f(x) = 0.75 + 0.373 \cos x + 0.89 \cos 2x + 1.005 \sin x - 0.109 \sin 2x.$$

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(2) Find Fourier series for the following function upto first harmonic.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	10	12	15	20	17	11	10

Soln:

$$N = 6, \quad l = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

x	y	$y \cos x$	$y \sin x$
0	10	10	0
$\pi/3$	12	6	10.392
$2\pi/3$	15	-7.5	12.99
π	20	-20	0
$4\pi/3$	17	-8.5	-14.72
$5\pi/3$	11	$+5.5$	-9.526
	85	-14.5	-0.866

$a_0 = 28.33$

$a_1 = -4.833$

$b_1 = -0.289$

$a_2 = 0.834$

$b_2 = 0.866$

$f(x) = 14.165 - 4.833\cos x - 0.289\sin x$

(3) Compute the first harmonic of Fourier series of $f(x)$ from the following data:

x :	0	30	60	90	120	150	180	210
y :	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25
	240	270	300	330	360			
	1.3	1.52	1.76	2.0	1.8			

Soln:

$2l = 2\pi \Rightarrow l = \pi$

$$y = 1.26 + 0.04\cos x + 0.53\cos 2x - 0.1\cos 3x - 0.63\sin x - 0.23\sin 2x + 0.085\sin 3x.$$

(4) Find the Fourier Series as far as the second harmonic to represent the function given in the following data:

class

x :	0	1	2	3	4	5
$f(x)$:	9	18	24	28	26	20

Soln:

$N = 6$

$2l = 6 \Rightarrow \boxed{l = 3}$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_1\sin\frac{\pi x}{3} + b_2\sin\frac{2\pi x}{3}$$

x	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	0	9	0	9	0
1	18	9	15.7	−9	15.6
2	24	−12	20.9	−24	0
3	28	−28	0	+28	0
4	26	−13	−22.6	−13	22.6
5	20	10	−17.4	−10	−17.4
	125	−25	−3.4	−19	20.8

$$f(x) = \frac{41.66}{2} - 8.33\cos\left(\frac{\pi x}{3}\right) - 6.33\cos\left(\frac{2\pi x}{3}\right)$$
$$- 1.13\sin\left(\frac{\pi x}{3}\right) + 0.009\sin\left(\frac{2\pi x}{3}\right)$$

(5) Find the half range Fourier sine series for the following values upto third harmonic.

x : 0 $\pi/6$ $2\pi/6$ $3\pi/6$ $4\pi/6$ $5\pi/6$ π

$f(x)$: 2.34 2.2 1.6 0.83 0.51 0.88 0.19

Soln:

$N = 6$, $l = \pi$,

$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$

x	y	$y\sin x$	$y\sin 2x$	$y\sin 3x$	
0	2.34	0	0	0	
$\pi/6$	2.2	1.1	1.91	2.2	$b_1 = 1.40$
$2\pi/6$	1.6	1.392	1.392	0	$b_2 = 1.207$
$3\pi/6$	0.83	0.83	0	−0.83	$b_3 = 0.75$
$4\pi/6$	0.51	0.44	−0.44	0	
$5\pi/6$	0.88	0.44	0.76	0.88	