Volume of triple integral :

$$\boxed{V = \iiint dx\,dy\,dz} \text{ (or) } \boxed{\iiint dz\,dy\,dx}$$

① Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Soln:

$V = 8 \times$ volume of the first octant.

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a.

$x^2+y^2+z^2=a^2$

$z^2=a^2-x^2-y^2$

$z=\pm\sqrt{a^2-x^2-y^2}$

$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\pm\sqrt{a^2-x^2}$

$x^2=a^2$

$x=\pm a$

$$V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-a^2+x^2} + \frac{a^2-x^2}{2}\sin^{-1}(1) - 0\right] dx$$

$$= 8\int_0^a \left(0 + \frac{a^2-x^2}{2}\,\frac{\pi}{2}\right) dx$$

$$\boxed{V = \frac{4}{3}\pi a^3}$$

(2) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Soln:

$$\text{Volume} = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{K^2-y^2}\,dy\,dx \qquad K = b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{K^2-y^2}+\frac{K^2}{2}\sin^{-1}\left(\frac{y}{K}\right)\right]_0^K dx$$

$$= \frac{8c}{b}\int_0^a \frac{K^2}{2}\cdot\frac{\pi}{2}\,dx$$

$\boxed{V = \frac{4}{3}\pi abc}$ cubic units.

③ Find the volume of the tetrahedron bounded by the coordinate planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Soln:

$$V = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\, dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \left(y - \frac{x}{a}y - \frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right) b \left(1-\frac{x}{a}\right) - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right] dx$$

$$= \frac{cb}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{-3(1/a)}\right]_0^a$$

$$V = \frac{abc}{6} \text{ cubic units}$$