Double integration in Cartesian Coordinates:

① Evaluate $\int_0^1 \int_1^2 x(x+y)\,dx\,dy$

$$= \int_0^1 \left[\frac{x^3}{3} + y\frac{x^2}{2}\right]_1^2 dy$$

$$= \int_0^1 \left(\frac{8}{3} - \frac{1}{3} + \frac{y}{2}(4-1)\right) dy$$

$$= \int_0^1 \left(\frac{7}{3} + 2y - \frac{y}{2}\right) dy = \left[\frac{7}{3}y + 2\frac{y^2}{2} - \frac{1}{2}\frac{y^2}{2}\right]_0^1$$

$$= \frac{7}{3} + 1 - \frac{1}{4} = 37/12$$

② Evaluate $\int_0^3 \int_0^2 e^{x+y}\,dy\,dx$

$$= \int_0^3 \left[e^{x+y}\right]_0^2 dx = \int_0^3 (e^{x+2} - e^{x+0})\,dx$$

$$= \left[e^{x+2} - e^x\right]_0^3 = e^5 - e^3 - e^2 + 1$$

③ Evaluate $\int_2^b \int_2^a \frac{dx\,dy}{xy}$

$$= \int_2^b \frac{dy}{y} (\log x)_2^a = (\log a - \log 2) \int_2^b \frac{dy}{y}$$

$$= (\log a - \log 2)\,(\log y)_2^b$$

$$= \log\left(\frac{a}{2}\right) \log\left(b/2\right)$$

(4) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$

$$= \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - 0$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$

(5) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ $\quad\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right.$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2}\right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]\right] dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{2})\right]$$