UNIT - 3

Analytical Function

Definition :

*A Function $f(z)$ is said to be analytical in a region R of z plane if it is analytic at every point of R.

Cauchy - Riemann Equation (CR Equation).

*Necessary condition for a function $f(z)$ to be analytic

1. $f(z) = u + iv$ is analytic in the u, v function of x, y Region R then,

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

(ii) $u_x = v_y$ and $u_y = -v_x$ (or)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

*Sufficient condition for a function $f(z)$ to be analytic.

2) A function $f(z) = u + iv$ is analytic in region R, if,

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists

(ii) $u_x = v_y$ and $u_y = -v_x$ (or)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note :

$z = x + iy$, $f(z) = w = u + iv$ where u and v are functions of x and y

Example : 1

Test whether the function $f(z) = \bar{z}$ is analytic or not.

Soln: Given,

$$f(z) = \bar{z}$$

$$f(z) = u + iv \quad , \quad z = x + iy .$$

$$u + iv = \overline{(x + iy)}$$

$$u + iv = x - iy$$

Equate Real part and imaginary part

$u = x$	, $v = -y$
$\dfrac{\partial u}{\partial x} = 1$	$\dfrac{\partial v}{\partial x} = 0$
$\dfrac{\partial u}{\partial y} = 0$	$\dfrac{\partial v}{\partial y} = -1$

The CR equation are $u_x = v_y$ and $u_y = -v_x$

$$1 \neq -1 \quad \text{and} \quad 0 = 0$$

CR equations are not satisfied

$\therefore F(x)$ is not analytic.

Example : 2

Check whether the function $w = 2xy + i(x^2 - y^2)$ is analytic or not ?

Soln: Given,

$$w = f(z)$$

$$f(z) = 2xy + i(x^2 - y^2)$$

$$u + iv = 2xy + i(x^2 - y^2)$$

Equate real part and imaginary part.

$u = 2xy$	, $v = x^2 - y^2$
$\dfrac{\partial u}{\partial x} = 2y$	, $\dfrac{\partial v}{\partial x} = 2x$
$\dfrac{\partial u}{\partial y} = 2x$	$\dfrac{\partial v}{\partial y} = -2y$

The CR equation are $U_x = V_y$ and $U_y = -V_x$

$2y \neq -2y$ and $2x \neq -2x$

$\therefore$ CR equation is not satisfied

$\therefore f(z)$ is not analytic.

Example : 3

Let $f(z) = z^3$ is analytic ? Justify

Soln: Given, $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$i^2 = -1$

$i^3 = i^2 \cdot i$

$= (-1) \cdot i$

$i^3 = -i$

$$f(z) = z^3$$
$$u + iv = z^3$$
$$u + iv = (x + iy)^3$$
$$u + iv = x^3 + (iy)^3 + 3(x)^2(iy) + 3x(iy)^2$$
$$u + iv = x^3 - iy^3 + 3x^2yi - 3xy^2$$

Equate real part and imaginary part

$$u = x^3 - 3xy^2, \quad v = -y^3 + 3x^2y$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = -3y^2 + 3x^2 = 3x^2 - 3y^2$$

The CR equation are $U_x = V_y$ and $U_y = -V_x$.

$3x^2 - 3y^2 = 3x^2 - 3y^2$ and

$-6xy = -6xy$.

$\therefore$ CR equation is satisfied

$\therefore f(z)$ is analytic.

$\therefore$ Hence proved.

Example : 4

Find the constant a, b, c and $f(z) = 2x + ay + i(bx + cy)$ is analytic

Soln:

$u + iv = x + ay + i(bx + cy)$

Equating real part and imaginary part.

$u = x + ay$ $\qquad\qquad$ $v = bx + cy.$

$\frac{\partial u}{\partial x} = 1$ $\qquad\qquad$ $\frac{\partial v}{\partial x} = b$

$\frac{\partial u}{\partial y} = a$ $\qquad\qquad$ $\frac{\partial v}{\partial y} = c.$

The CR equation are $U_x = V_y$ and $U_y = -V_x$.

$1 = c$ and $a = -b$

CR equation is satisfied

They said $f(z)$ is analytic

The value of

$a = -b$

$c = 1$

$b = -a$