UNIT IV COMPLEX INTEGRATION

Part-A

Problem 1 Evaluate $\int_C \frac{z}{(z-1)^3}\, dz$ where C is $|z|=2$ using Cauchy's integral formula

Solution:

Given $\int_C \frac{z}{(z-1)^3}\, dz$

Here $f(z)=z$, $a=1$ lies inside $|z|=2$

$$\therefore \int_C \frac{zdz}{(z-1)^3} = \frac{2\pi i}{2!} f''(1)$$

$$= \pi i[0] \because f''(1)=0$$

$$\therefore \int_C \frac{zdz}{(z-1)^3} = 0.$$

Problem 2 State Cauchy's Integral formula

Solution:

If $f(z)$ is analytic inside and on a closed curve C that encloses a simply connected region R and if $'a'$ is any point in R, then $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz$.

Problem 3 Evaluate $\int_C e^{\frac{1}{z}} dz$ where C is $|z-2|=1$.

Solution:

$e^{\frac{1}{z}}$ is analytic inside and on C.

Hence by Cauchy's integral theorem $\int_C e^{\frac{1}{z}} dz = 0$

Problem 4 Classify the singularities of $f(z)=\frac{e^{\frac{1}{z}}}{(z-a)^2}$.

Solution:

Poles of $f(z)$ are obtained by equating the denominator to zero.

i.e., $(z-a)^2=0$, $z=a$ is a pole of order 2

The principal part of the Laurent's expansion of $e^{1/z}$ about z = 0 contains infinite number terms. Therefore there is an essential singularity at z = 0.

Problem 5 Calculate the residue of $f(z)=\dfrac{1-e^{2z}}{z^3}$ at the poles.

Solution:

Given $f(z)=\dfrac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=0} = \underset{z\to 0}{Lt}\ \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-0)^3\frac{1-e^{2z}}{z^3}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ \frac{d^2}{dz^2}\left[1-e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ \frac{d}{dz}\left[-2e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ -4e^{2z}$$

$$=\frac{1}{2}(-4)=-2.$$

Problem 6 Evaluate $\int_C \dfrac{\cos\pi z}{z-1}dz$ if C is $|z|=2$.

Solution:

We know that, Cauchy Integral formula is $\int_C \dfrac{f(z)}{z-a}\,dz = 2\pi i f(a)$ if 'a' lies inside C

$\int_C \dfrac{\cos\pi z}{z-1}dz$, Here $f(z)=\cos\pi z$

$\therefore z=1$ lies inside C

$\therefore f(1)=\cos\pi(1)=-1.$

$\therefore \int_C \dfrac{\cos\pi z}{z-1}dz = 2\pi i(-1) = -2\pi i$.

Problem 7 Define Removable singularity

Solution:

A singular point $z=z_0$ is called a removable singularity of $f(z)$ is $\underset{z\to z_0}{Lt} f(z)$ exists finitely

Example: For $f(z)=\frac{\sin z}{z}$, z = 0 is a removable singularity since $\underset{z\to 0}{Lt} f(z)=1$

Problem 8
Test for singularity of $\frac{1}{z^2+1}$ and hence find corresponding residues.

Solution:

Let $f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$

Here $z=-i$ is a simple pole

$z=i$ is a simple pole

$$Res(z=i)=\underset{z\to i}{Lt}(z-i)\frac{1}{(z+i)(z-i)}$$

$$=\underset{z\to i}{Lt}\frac{1}{(z+i)}=\frac{1}{2i}$$

$$Res(z=-i)=\underset{z\to -i}{Lt}(z+i)\frac{1}{(z+i)(z-i)}=\frac{1}{-2i}.$$

Problem 9
What is the value of $\int_C e^z dz$ where C is $|z|=1$.

Solution:

Put $z=e^{i\theta}$

$dz=ie^{i\theta}d\theta$

$$\int_C e^z dz=\int_0^{2\pi} e^{e^{i\theta}} ie^{i\theta}d\theta ....(1)$$

Put $t=e^{i\theta} \Rightarrow dt=e^{i\theta}d\theta$

When $\theta=0,\ t=1,\ \theta=2\pi,\ ssss\, t=1$

$$\therefore (1)\Rightarrow \int_C e^z dz=\int_1^1 e^t dt=\left[e^t\right]_1^1=0$$

Problem 10
Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$, where $|z|=\frac{1}{2}$.

Solution:

Given $\int_C \frac{3z^2+7z+1}{z+1}dz$

Here $f(z)=3z^2+7z+1$

$z=-1$ lies outside $|z|=\frac{1}{2}$

Here $\int_C \frac{3z^2+72+1}{z+1}dz = 0$.(By Cauchy Theorem)

Problem 11 State Cauchy's residue theorem

Solution:

If $f(z)$ be analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2, ..., z_n$ inside C then $\int_C f(z)dz = 2\pi i \times \left[\text{sum of the residue of } f(z) \text{ at } z_1, z_2, ..., z_n \right]$.

Problem 12 Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$

Here $z = -1$ is a pole of order 2

$$\left[Resf(z) \right]_{z=-1} = \underset{z \to -1}{Lt} \frac{1}{1!} \frac{d}{dz} (z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \underset{z \to -1}{Lt} 2e^{2z} = 2e^{-2}.$$

Problem 13 Using Cauchy integral formula evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$, where $|z| = \frac{3}{2}$

Solution:

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = \int_C \frac{-\cos \pi z^2}{z-1}dz + \int_C \frac{\cos \pi z^2}{(z-2)} dz$$

$$\left[\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{z-2}, \quad A = -1 \quad B = 1 \right]$$

Here $f(z) = \cos \pi z^2$

$z = 1$ lies inside $|z| = \frac{3}{2}$

$z = 2$ lies outside $|z| = \frac{3}{2}$

Hence by Cauchy integral formula

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(z)$$

$$= -2\pi i(-1)$$

$$= 2\pi i \qquad [\therefore \, f(z) = \cos \pi z, \; f(1) = \cos \pi = -1]$$

Problem 14 State Laurent's series

Solution:

If C_1 and C_2 are two concentric circles with centres at $z = a$ and radii r_1 and r_2 $(r_1 < r_2)$ and if $f(z)$ is analytic on C_1 and C_2 and throughout the annular region R between them, then at each point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$, $n = 0, 1, 2, \ldots$, $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z-a)^{-n+1}}$, $n = 1, 2, 3, \ldots$

Problem 15 Find the zeros of $\dfrac{z^3 - 1}{z^3 + 1}$.

Solution:

The zeros of $f(z)$ are given by $f(z) = 0$, $\dfrac{z^3 - 1}{z^3 + 1} = 0$

i.e., $z^3 - 1 = 0$, $z = (1)^{\frac{1}{3}}$

$z = 1, \; w, w^2$ (Cubic roots of unity)

Part-B

Problem 1 Using Cauchy integral formula evaluate $\displaystyle\int_C \frac{dz}{(z+1)^2 (z-2)}$ where C the circle $|z| = \dfrac{3}{2}$.

Solution:

Here $z = -1$ is a pole lies inside the circle

$z = 2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2 (z-2)} = \int \frac{\frac{1}{z-2}}{(z+1)^2} dz$$

Here $\mathrm{f}(z) = \dfrac{1}{z-2}$

$f'(z) = -\frac{1}{(z-2)^2}$

Hence by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2 (z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2} dz$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i \left[\frac{-1}{(-1-2)^2} \right] \quad \left(\therefore f'|z| = \frac{-1}{(z-2)^2} \right)$$

$$= 2\pi i \left[\frac{-1}{9} \right]$$

$$= \frac{-2}{9} \pi i .$$

Problem 2

Evaluate $\int_C \frac{z-2}{z(z-1)} dz$ where C is the circle $|z| = 3$.

Solution:

W.K.T $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \, dz$

Given $\int_C \frac{z-2}{z(z-1)} dz$ Here $z = 0$, $z = 1$ lies inside the circle

Also $f(z) = z - 2$

Now $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

Put $z = 0 \Rightarrow A = -1$

$z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{z-2}{z(z-1)} dz = -\int_C \frac{z-2}{z} dz + \int_C \frac{z-2}{z-1} dz$$

$$= -2\pi i \; f(0) + 2\pi i \; f(1)$$

$$= 2\pi i [f(1) - f(0)]$$

$$= 2\pi i [-1 - (-2)]$$

$$= 2\pi i [2 - 1] = 2\pi i .$$

Problem 3

Find the Laurent's Series expansion of the function $\frac{z-1}{(z+2)(z+3)}$, valid in the region $2<|z|<3$.

Solution:

Let $f(z)=\frac{z-1}{(z+2)(z+3)}$

$\frac{z-1}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$

$z-1=A(z+3)+B(z+2)$

Put $z=-2$

$-2-1=A(-2+3)+0$

$A=3$

Put $z=-3$

$-3-1=A(0)+B(-3+2)$

$-4=-B$

$B=4$

$\therefore f(z)=\frac{-3}{z+2}+\frac{4}{z+3}$

Given region is $2<|z|<3$

$2<|z|$ and $|z|<3$

$\left|\frac{2}{z}\right|<1$ and $\left|\frac{z}{3}\right|<1$

$$\therefore f(z)=\frac{-3}{z\left(1+\frac{2}{z}\right)}+\frac{4}{3\left(1+\frac{z}{3}\right)}$$

$$=\frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1}+\frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{-3}{z}\left[\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-....\right)\right]+\frac{4}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-...\right]$$

Problem 4

Expand $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ valid in $1<|z+1|<3$

Solution:

Given $f(z)=\frac{7z-2}{z(z-2)(z+1)}$

$$f(z)=\frac{7z-2}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$$

$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

Put $z=2$

$B=2$

Put $z=0$

$A=1$

Put $z=-1$

$C=-3$

$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

Given region is $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow 1 < |u| < 3$

$z = u-1 \Rightarrow 1 < |u|$ & $|u| < 3$

$\Rightarrow \frac{1}{|u|} < 1$ & $\left|\frac{u}{3}\right| < 1$

$$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u(1-\frac{1}{u})} + \frac{2}{-3(1-\frac{u}{3})} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\dots\right] - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\dots\right) - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\dots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\dots\right] - \frac{3}{z+1}$$

$$\therefore f(z) = -\frac{2}{z+1} + \sum_{n=2}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n}.$$

Problem 5

Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Taylor series valid in the region $|z| < 2$.

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

Now $(z+2)(z+3) = z^2+5z+6$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

Now $\dfrac{-5z-7}{(z+2)(z+3)}=\dfrac{A}{z+2}+\dfrac{B}{z+3}$

$-5z-7=A(z+3)+B(z+2)$

Put $z=-2$

$A=3$

Put $z=-3$

$B=-8$

$\therefore f(z)=1+\dfrac{3}{z+2}-\dfrac{8}{z+3}$

Given $|z|<2$

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^{2}+\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}+\ldots\right)$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z}{2}\right)^{n}-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z}{3}\right)^{n}$$

$$f(z)=1+\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right]z^{n}.$$

Problem 6 Using Cauchy Integral formula Evaluate $\displaystyle\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$ where C is circle $|z|=1$.

Solution:

Here $f(z)=\sin^6 z$

$f'(z)=6\sin^5\cos z$

$f''(z)=6\left[-\sin^6 z+\cos^2 z.5\sin^4 z\right]$

Here $a=\dfrac{\pi}{6}$, clearly $a=\dfrac{\pi}{6}$ lies inside the circle $|z|=1$

By Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^3}=\frac{2\pi i}{2!}f''(a)$$

$$\therefore \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \pi i 6\left[-\sin^6\left(\frac{\pi}{6}\right)+5\cos^2\left(\frac{\pi}{6}\right)\sin^4\frac{\pi}{6}\right]$$

$$= 6\pi i\left[-\frac{1}{64}+\frac{5}{16}\times\frac{3}{4}\right]$$

$$= 6\pi i\left[-\frac{1}{64}+\frac{15}{64}\right]$$

$$= 6\pi i\left[\frac{15-1}{64}\right] = \frac{21\pi i}{16}$$

Problem 7 Expand $f(z)=\sin z$ into a Taylor's series about $z=\frac{\pi}{4}$.

Solution:

Given $f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$

$\therefore f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$f'\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$f''\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$f'''\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

W.K.T Taylor's series of $f(z)$ at $z=a$ is

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots$$

$$f(z)=f\left(\frac{\pi}{4}\right)+\frac{z-\frac{\pi}{4}}{1!}f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\ldots$$

$$= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \left(\frac{z - \frac{\pi}{4}}{2}\right)^2 \left(\frac{1}{\sqrt{2}}\right) + \ldots$$

Problem 8 Evaluate $\int_C \frac{z \sec z}{\left(1-z^2\right)} dz$ where C is the ellipse $4x^2 + 9y^2 = 9$, using Cauchy's residue theorem.

Solution:

Equation of ellipse is

$$4x^2 + 9y^2 = 9$$

$$\frac{x^2}{9/4} + \frac{y^2}{1} = 1$$

i.e., $\frac{x^2}{\left(3/2\right)^2} + \frac{y^2}{1} = 1$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the x axis at $\pm\frac{3}{2}$ and the y axis at ± 1

Given $f(z) = \frac{z \sec z}{1-z^2}$

$$= \frac{z}{(1+z)(1-z)\cos z}$$

The poles are the solutions of $(1+z)(1-z)\cos z = 0$

i.e., $z = -1,\ z = 1$ are simple poles and $z = (2n+1)\frac{\pi}{2}$

Out of these poles $z \pm 1$ lies inside the ellipse

$z = \pm\frac{\pi}{4},\ \pm 3\frac{\pi}{4}$ lies outside the ellipse

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \underset{z \to 1}{Lt} (z-1)\frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to 1}{Lt} \frac{-z}{(1+z)\cos z} = \frac{-1}{2\cos 1}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \underset{z \to -1}{Lt} (z+1)\frac{z}{(1+z)(1-z)\cos z}$$

$$= \underset{z \to -1}{Lt} \frac{z}{(1-z)\cos z}$$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[\text{sum of the residues }\right]$$

$$= 2\pi i\left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i\left[\sec 1\right].$$

Problem 9 Using Cauchy integral formula evaluate (i) $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1-i| = 2$ (ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, C is the circle $|z| = \frac{3}{2}$.

Solution:

(i) Given $|z+1-i| = 2$

$|z-(-1+i)| = 2$ is a circle whose centre is $-1+i$ and radius 2.

i.e., centre $(-1,1)$ and radius 2

$z^2+2z+5 = \left[z-(-1+2i)\right]\left[z-(-1-2i\right]$

$-1+2i$ i.e., $(-1,2)$ lies inside the C

$-1-2i$ i.e., $(-1,-2)$ lies out side the C

$$\left[\therefore z^2+2z+5 = 0 \Rightarrow z = -2 \pm \sqrt{\frac{4-20}{2}},\ z = -1 \pm 2i\right]$$

$$\therefore \int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

$$= \int_C \frac{\frac{z+4}{\left[z-(-1-2i)\right]}}{z-(-1+2i)}\,dz$$

Hence $f(z) = \frac{z+4}{\left[z-(-1-2i)\right]}$

Here by Cauchy integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\ f(a)$$

$$\int_C \frac{z+4}{z^2+2z+5} = 2\pi i\ f(-1+2i)$$

$$= 2\pi i\left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}[3+2i].$$

(ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$

$z=0,\ z=1$ lie inside the circle $|z| = \frac{3}{2}$

$z=2$ lies outside the circle

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$

$4-3z = A(z-1)(z-2) + B(z)(z-2) + C(z)(z-1)$

Put $z=0$

$4 = 4A$

$A = 1$

Put $z=1$

$B=-1$

Put $z=2$

$C=-1$

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$

$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_C \frac{2}{z}\,dz - \int_C \frac{1}{z-1}dz - \int_C \frac{1}{z-2}dz$

$\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!}\, f^{n}(0)$

$$= 2\,[2\pi i\, f(0)] - 2\pi i\ f(1) - 0$$

$$= 4\pi i\ f(0) - 2\pi i\ f(1)$$

$$= 4\pi i\ (1) - 2\pi i\ (1)$$

$$= 2\pi i \qquad (\therefore f(0)=1\ f(1)=1)$$

Problem 10

Using Cauchy's integral formula evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where C is circle $|z-i| = 2$

Solution:

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

Given $|z-i| = 2$, centre $(0,1)$, radius 2

$\therefore z=-2i$ lies outside the circle

$z = 2i$ lies inside the circle

$$\therefore \int_C \frac{dz}{\left(z^2+4\right)^2} = \int_c \frac{\frac{1}{\left(z+2i\right)^2}}{\left(z-2i\right)^2}\,dz$$

Here $f\left(z\right) = \dfrac{1}{\left(z+2i\right)^2}$

$$f'\left(z\right) = \frac{-2}{\left(z+2i\right)^3}$$

$$f'\left(2i\right) = -\frac{2}{\left(2i+2i\right)^3} = -\frac{2}{\left(4i\right)^3}$$

$$= -\frac{2i}{64} = -\frac{i}{32}$$

Hence by Cauchy Integral Formula

$$\int_C \frac{f\left(z\right)}{\left(z-a\right)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n\left(a\right)$$

$$\int_C \frac{f\left(z\right)}{\left(z^2+4\right)^2} = \frac{2\pi i}{1!} f'\left(2i\right) = \frac{\pi}{16}.$$

Problem 11

Find the Laurent's series which represents the function $\dfrac{z}{\left(z+1\right)\left(z+2\right)}$ in (i) $\left|z\right| > 2$ (ii) $\left|z+1\right| < 1$

Solution:

(i). Let $f\left(z\right) = \dfrac{z}{\left(z+1\right)\left(z+2\right)}$

Now $\dfrac{z}{\left(z+1\right)\left(z+2\right)} = \dfrac{A}{z+1} + \dfrac{B}{z+2}$

$z = A\left(z+2\right) + B\left(z+1\right)$

Put $z = -1$

$A = -1$

Put $z = -2$

$B = 1$

$$\therefore f\left(z\right) = \frac{-1}{z+1} + \frac{2}{z+2}$$

Given $\left|z\right| > 2,\ 2 < \left|z\right|$ i.e., $\left|\dfrac{2}{z}\right| < 1 \Rightarrow \dfrac{1}{\left|z\right|} < 1$

$$\therefore f\left(z\right) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{z\left(1+\frac{1}{z}\right)} + \frac{2}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{-1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1+\frac{2}{z}\right)^{-1}$$

(ii). $|z+1| < 1$

Let $u = z+1$

i.e., $|u| < 1$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{u} + \frac{2}{1+u}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}$$

$$= \frac{-1}{u} + 2(1-u+u^2-\ldots)$$

$$= \frac{-1}{1+z} + 2\left[1-(1+z)+(1+z)^2-\ldots\right]$$

Problem 12

Prove that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1-a^2}$, given $a^2 < 1$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1}$

Put $z = e^{i\theta}$

Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore\ I = \int_C \frac{dz/iz}{a^2 - a\left(z+\frac{1}{z}\right)+1} \text{ where C is } |z| = 1.$$

$$= \frac{1}{ai}\int_C \frac{dz}{\left(a+\frac{1}{a}\right)z - z^2 - 1}$$

$$= \frac{i}{a}\int_C \frac{dz}{z^2 - \left(a+\frac{1}{a}\right)z + 1}$$

$$= \int_C f(z)dz \text{ where } f(z) = \left(\frac{i}{a}\right)\frac{1}{z^2 - \left(a+\frac{1}{a}\right)z + 1}$$

$$= \left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

The singularities of f(z) are simple poles at a and $\frac{1}{a}$. $a^2 < 1$ implies $|a| < 1$ and $\frac{1}{|a|} > 1$

$\therefore$ The pole that lies inside C is z = a.

$$\text{Res}[f(z);\ a] = \lim_{z\to a}(z-a).\left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \left(\frac{i}{a}\right)\frac{1}{\left(a-\frac{1}{a}\right)}$$

$$= \frac{i}{a^2-1}$$

Hence $I = 2\pi i.\dfrac{i}{a^2-1} = \dfrac{2\pi}{1-a^2}$

Problem 13

Show that $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

Solution: Let $\displaystyle I = \int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta}$

Put $z = e^{i\theta}$

Then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$

$$I = \text{Real Part of } \int_0^{2\pi}\frac{e^{i2\theta}.d\theta}{5+4\cos\theta}$$

$$= \text{Real Part of } \int_C \frac{z^2.\,{}^{dz}\!/_{iz}}{5+2\left(z+\frac{1}{z}\right)} \text{ where C is } |z| = 1.$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{z^2+\frac{5}{2}z+1}$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \text{Real Part of } \int_C f(z)dz \text{ where } f(z) = \frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$z = -\frac{1}{2}$ and $z = -2$ are simple poles of f(z).

$z = -\frac{1}{2}$ lies inside C.

$$\text{Res}\left[f(z); -\tfrac{1}{2}\right] = \lim_{z\to -\frac{1}{2}}\left(z+\tfrac{1}{2}\right).\frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{2i}.\frac{1/4}{3/2} = \frac{1}{12i}$$

$\therefore$ I = Real Part of $2\pi i.\dfrac{1}{12i}$

= Real Part of $\pi/6$

= $\pi/6$.

Problem 14

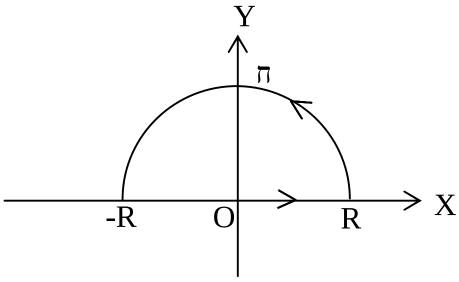
Prove that $\int_0^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

Solution:

Let $\int_C \phi(z)\,dz = \int_C \frac{dz}{\left(z^2+1\right)^2}$

Where $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$

Here C is the semicircle Γ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz$......(1)

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$ is the solution of $\left(z^2+1\right)^2 = 0$

i.e., $(z+i)^2(z-i)^2 = 0$

i.e., the poles are $z = i,\ z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $\left[Res\,\phi(z)\right]_{z=i}$ = $\underset{z \to i}{Lt} \frac{1}{1!} \frac{d}{dz} (z-i)^2 \phi(z)$

$$= \underset{z \to i}{Lt} \frac{1}{1!}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right]$$

$$= \underset{z \to i}{Lt} \frac{1}{1!} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1=(z+i)(z-i)$$

$$= \underset{z \to i}{Lt} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\, dz = 2\pi i\left[Sum\ of\ \ residues\ of\ \phi(z) \text{at } its\ \ poles\ \ which\ lies\, in\, C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2} ....(2)$$

Let $R \to \infty$, then $|z| \to \infty$ so that $\phi(z) = 0$

$$\therefore \underset{|z| \to \infty}{Lt} \int_\Gamma \phi(z) dz = 0.........(3)$$

Sub (2) and (3) in (1)

$$\int_C \phi(z) dz = \int_{-\infty}^{\infty} \phi(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Problem 15 Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\, dx$

Solution:

$$2\int_0^{\infty} \frac{x\sin x}{x^2+a^2} dx = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} dx$$

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{z\sin z}{z^2+a^2} dz$$

$$= \frac{1}{2} I ....(1)$$

Now $z \sin z$ is the imaginary part of ze^{iz}

$$\therefore \text{I} = \int_{-\infty}^{\infty} \frac{z \sin z}{z^2 + a^2} dz$$

$$= \text{I.P.} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz$$

Let $\phi(z) = \dfrac{z e^{iz}}{z^2 + a^2} = \dfrac{ze^{iz}}{(z+ia)(z-ia)}$

The poles are $z = -ia$, $z = ia$

Now the poles $z = ia$ lies in the upper half – plane

But $z = -ia$ lies in the lower half – plane.

Hence

$$\left[Res\phi(z)\right]_{z=ia} = \underset{z \to ia}{Lt} (z - ia) \frac{ze^{iz}}{(z+ia)(z-ia)}$$

$$= \underset{z \to ia}{Lt} \frac{ze^{iz}}{(z+ia)}$$

$$= \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \left[\text{Sum of the residues at each poles in the upper half plane}\right]$$

$$= 2\pi i \left[\frac{e^{-a}}{2}\right]$$

$$= \pi i e^{-a}$$

I=I.P. of $\displaystyle\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz$

= I.P. of $\left(\pi i e^{-a}\right)$

$\text{I} = \pi e^{-a}$...........(2)

Sub (2) in (1)

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} x = \frac{1}{2} \pi e^{-a}$$