Generating Functions:

The generating function for the sequence 'S' with terms $a_0, a_1, \ldots, a_n$ of real numbers is the infinite sum.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

Working Rule:

Step 1: Rewrite the recurrence eqn., with RHS = 0

Step 2: Multiply the eqn. obtained in step 1 by x^n & summing from $(0 \text{ to } \infty)$ or $(1 \text{ to } \infty)$ or $(2 \text{ to } \infty)$

Step 3: Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and write $G(x)$ as a function of x.

Step 4: Decompose $G(x)$ into partial fraction.

Step 5: Express a_n as the co-efficient of x^n is $G(x)$

Note:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

1. Use the method of generating function to solve the recurrence eqn. $a_n = 3a_{n-1} + 1,\ n \geq 1$ given $a_0 = 1$.

Givn. $a_n = 3a_{n-1} + 1$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow a_n - 3a_{n-1} - 1 = 0$

multiply by x^n & taking summation

$$\sum_{n=1}^{\infty} x^n a_n - 3\sum_{n=1}^{\infty} x^n a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$\sum_{n=1}^{\infty} x^n a_n - 3\sum_{n=1}^{\infty} x \frac{x^n}{x} a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$\sum_{n=1}^{\infty} x^n a_n - 3x\sum_{n=1}^{\infty} x^{n-1} a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$[G(x) - a_0] - 3x\,G(x) - [x + x^2 + x^3 + \cdots] = 0$$

$$G(x)[1-3x] - a_0 - [(1-x)^{-1} - 1] = 0$$

$$G(x)[1-3x] - 1 - (1-x)^{-1} + 1 = 0$$

$$G(x)(1-3x) = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)(1-3x)} \longrightarrow (1)$$

using partial fraction,

$$\frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$1 = A(1-3x) + B(1-x)$$

when $x = 1$, $A = -1/2$

$x = 0$, $B = 3/2$

$$\therefore \frac{1}{(1-x)(1-3x)} = \frac{-1/2}{1-x} + \frac{3/2}{1-3x}$$

$$(1) \Rightarrow \quad G(x) = -\frac{1}{2}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-3x)}$$

$$= -\frac{1}{2}(1-x)^{-1} + \frac{3}{2}(1-3x)^{-1}$$

$$G(x) = -\frac{1}{2}[1+x+x^2+\dots] + \frac{3}{2}[1+3x+(3x)^2+\dots]$$

$$\sum_{n=0}^{\infty} a_n x^n = -\frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty}(3x)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}(1)^n x^n + \frac{3}{2}\sum_{n=0}^{\infty}(3)^n x^n$$

$$\therefore a_n = -\frac{1}{2}(1)^n + \frac{3}{2}(3)^n \quad [\because a_n - \text{the coeffi. of } x^n]$$

2. Using generating functions, solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$ gvn. that $a_0 = 2$, $a_1 = 1$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Gvn. $a_{n+2} - 2a_{n+1} + a_n = 2^n$ with $a_0 = 2$, $a_1 = 1$

multiply by x^n and taking summation

$$\sum_{n=0}^{\infty} x^n a_{n+2} - 2\sum_{n=0}^{\infty} x^n a_{n+1} + \sum_{n=0}^{\infty} x^n a_n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{x^2} a_{n+2} - 2\sum_{n=0}^{\infty}\frac{x^{n+1}}{x} a_{n+1} + G(x) = \sum_{n=0}^{\infty}(2x)^n$$

$$\frac{1}{x^2}[G(x) - a_0 - a_1 x] - \frac{2}{x}[G(x) - a_0] + G(x) = (1-2x)^{-1}$$

$$\frac{G(x)}{x^2} - \frac{2}{x^2} - \frac{x}{x^2} - \frac{2}{x}G(x) + \frac{4}{x} + G(x) = \frac{1}{1-2x}$$

$$G(x)\left[\frac{1}{x^2} - \frac{2}{x} + 1\right] + \frac{4}{x} - \frac{2}{x^2} - \frac{1}{x} = \frac{1}{1-2x}$$

$$G(x)\left[\frac{1-2x+x^2}{x^2}\right]+\frac{2}{x}-\frac{2}{x^2}=\frac{1}{1-2x}$$

$$G(x)\left[\frac{(x-1)^2}{x^2}\right]=\frac{1}{1-2x}+\frac{2-3x}{x^2}$$

$$=\frac{x^2+(1-2x)(2-3x)}{x^2(1-2x)}$$

$$=\frac{x^2+2-3x-4x+6x^2}{x^2(1-2x)}$$

$$G(x)=\frac{7x^2-7x+2}{x^2(1-2x)}\quad\frac{x^2}{(x-1)^2}$$

$$G(x)=\frac{7x^2-7x+2}{(1-2x)(x-1)^2}$$

By using partial fractions

$$\frac{7x^2-7x+2}{(1-2x)(x-1)^2}=\frac{A}{1-2x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\rightarrow(1)$$

$$=\frac{A(x-1)^2+B(1-2x)(x-1)+C(1-2x)}{(1-2x)(x-1)^2}$$

$$7x^2-7x+2=A(x-1)^2+B(1-2x)(x-1)+C(1-2x)$$

$x=1$, $\quad 2=C(1-2)$

$\quad -C=2 \Rightarrow C=-2$

$x=0$, $\quad 2=A-B+C \Rightarrow A-B=4$

$x=\frac{1}{2}$, $\quad \frac{7}{4}-\frac{7}{2}+2=\frac{A}{4}+0+0$

$$\frac{A}{4}=\frac{7-14+8}{4}=\frac{15-14}{4}$$

$$A=1$$

$$\therefore \quad B = A - 4$$
$$= 1 - 4$$
$$B = -3$$

Here $A = 1,\ B = -3,\ C = -2$

$$(1) \Rightarrow \quad G(x) = \frac{1}{1-2x} - \frac{3}{x-1} - \frac{2}{(x-1)^2}$$

$$= (1-2x)^{-1} + 3(1-x)^{-1} - 2(1-x)^{-2}$$

$$= 1 + 2x + (2x)^2 + (2x)^3 + \cdots + 3[1 + x + x^2 + x^3 + \cdots]$$
$$- 2[1 + 2x + 3x^2 + 4x^3 + \cdots]$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2x)^n + 3\sum_{n=0}^{\infty} (x)^n - 2\sum_{n=0}^{\infty} (n+1)x^n$$

$$a_n = (2)^n + 3(1)^n - 2(n+1)$$

- using generating function. Solve the recurrence relation $a_{n+2} - 5a_{n+1} - 6a_n = 2$, by using $a_0 = 1,\ a_1 = 2$.
- Solve the recurrence relation, $a_{n+2} - 4a_{n+1} + 3a_n = 0$, $a_0 = 2,\ a_1 = 4$ using generating function.
- Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $a_0 = 10,\ a_1 = 41, n \geq 2$ using generating function.