

Taylor's Series:  $1 - \frac{z}{1} + \frac{(1)z^2}{2!} + 0 =$

If  $f(z)$  is analytic inside a circle  $C$  with centre at  $z = a$ , then  $f(z)$  can be expressed as,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots +$$

$$\frac{(z-a)^n}{n!} f^n(a) + \dots$$

which is convergent at every point inside  $C$ . This is called Taylor's series of  $f(z)$  about  $z = a$ .

Note:

The Taylor's series of  $f(z)$  about the point  $z = 0$  is given by,

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0) + \dots$$

① Expand  $f(z) = \log(1+z)$  as Taylor's series about  $z = 0$  if  $|z| < 1$ .

Soln:

$$f(z) = \log(1+z)$$

$$f(0) = \log 1 = 0$$

$$f'(z) = \frac{1}{1+z}$$

$$f'(0) = \frac{1}{1} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2}$$

$$f''(0) = \frac{-1}{1} = -1$$

$$f'''(z) = \frac{2}{(1+z)^3}$$

$$f'''(0) = \frac{2}{1} = 2$$

Taylor's series about  $z=0$  is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \dots$$

$$= 0 + \frac{z}{1!} (1) + \frac{z^2}{2!} (-1) + \frac{z^3}{3!} (2) + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

If  $f(z)$  is analytic inside a circle of radius  $r$  and  $z=0$  is the centre of the circle, then  $f(z)$  can be expressed as

2) Expand  $f(z) = \cos z$  about  $z = \pi/3$  in Taylor's series.

Soln:

$$f(z) = \cos z$$

$$f(\pi/3) = \cos \pi/3 = 1/2$$

$$f'(z) = -\sin z$$

$$f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$$

$$f''(z) = -\cos z$$

$$f''(\pi/3) = -\cos \pi/3 = -1/2$$

$$f'''(z) = \sin z$$

$$f'''(\pi/3) = \sin \pi/3 = \sqrt{3}/2$$

Taylor's series is,

$$f(z) = f(\pi/3) + \frac{(z-\pi/3)}{1!} f'(\pi/3) + \frac{(z-\pi/3)^2}{2!} f''(\pi/3) + \dots$$

$$= \frac{1}{2} + \frac{(z-\pi/3)}{1!} \left(-\frac{\sqrt{3}}{2}\right) + \frac{(z-\pi/3)^2}{2!} \left(-\frac{1}{2}\right) + \frac{(z-\pi/3)^3}{3!} \left(\frac{\sqrt{3}}{2}\right) + \dots$$

$$= \frac{1}{2} - \frac{(z-\pi/3)}{1!} \left(\frac{\sqrt{3}}{2}\right) - \frac{(z-\pi/3)^2}{2!} \left(\frac{1}{2}\right) + \frac{(z-\pi/3)^3}{3!} \left(\frac{\sqrt{3}}{2}\right) + \dots$$

③ Expand  $f(z) = \sin z$  in a Taylor's Series about  $z = \pi/4$ .

Soln:

$$f(z) = \sin z$$

$$f'(z) = \cos z$$

$$f''(z) = -\sin z$$

$$f'''(z) = -\cos z$$

$$f(\pi/4) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(\pi/4) = \cos \frac{\pi}{4} = 1/\sqrt{2}$$

$$f''(\pi/4) = -\sin \frac{\pi}{4} = -1/\sqrt{2}$$

$$f'''(\pi/4) = -\cos \frac{\pi}{4} = -1/\sqrt{2}$$

Taylor's Series formula for  $z=a$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \dots$$

At  $z = \pi/4$

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \dots$$

$$f(z) = \frac{1}{\sqrt{2}} + (z-\frac{\pi}{4}) \frac{1}{\sqrt{2}} + \frac{(z-\frac{\pi}{4})^2}{2} \cdot \left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\frac{\pi}{4})^3}{6} \cdot \left(\frac{-1}{\sqrt{2}}\right) + \dots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}} \left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}} \left(z-\frac{\pi}{4}\right)^3 + \dots$$