UNIT - V

Laplace Transforms

Definition:

Let $f(t)$ be a function of t defined for all $t>0$. Then the Laplace Transform of $f(t)$, denoted by $L[f(t)]$ or $F(s)$, is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s),$$

provided that the integral exists.

s – parameter (real or complex)

s – +ve real number (generally)

Transforms of elementary functions.

1) $L[1] = \frac{1}{s}$, $s>0$.

Pf: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\infty e^{-st}(1).\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s}$$

$$= 0 - \left(\frac{1}{-s}\right)$$

$$= \frac{1}{s} \qquad [\because e^{-\infty} = 0 \text{ \& } e^0 = 1]$$

$$\therefore L[1] = \frac{1}{s}, \quad s>0.$$

(ii) $L[k] = k\,L[1] = \frac{k}{s}$

(iii) $L(0) = 0$

(iv) $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ $\left(\Gamma(n+1) = n! \text{ for non-ve integer}\right)$

$$= \frac{n!}{s^{n+1}}$$

Pf:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt$$

put $st = x \Rightarrow t = \frac{x}{s}$.

Diff. w.r.to x

$$s.\frac{dt}{dx} = 1$$

$$dx = s.dt$$

$$\Rightarrow dt = \frac{dx}{s}$$

when $t=0$; $x=0$

$t=\infty$; $x=\infty$.

$$\therefore L[t^n] = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n.\frac{dx}{s}$$

$$= \int_0^\infty e^{-x}.\frac{x^n}{s^{n+1}}.\,dx$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \cancel{\frac{1}{s^{n+1}}\left[\frac{e^{-x}}{-1}\right]_0^\infty}$$

since $\Gamma(n+1) = \int e^{-x} x^n\,dx$

$$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Note $\Gamma(1/2) = \sqrt{\pi}$

(v) $L[e^{at}] = \frac{1}{s-a}$, $s > a$

Pf: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$

$= \int_0^\infty e^{-st+at}\, dt$

$= \int_0^\infty e^{(-s+a)t}\, dt$

$= \int_0^\infty e^{-t(s-a)}\, dt$

$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$

$= \left[\frac{e^{-\infty}}{-(s-a)} - \frac{e^0}{-(s-a)}\right]$

$= 0 - \frac{1}{(-(s-a))}$

$= \frac{1}{s-a}$

$\therefore L[e^{at}] = \frac{1}{s-a}$.

6) $L[e^{-at}] = \frac{1}{s+a}$.

7) $L[\sin at] = \frac{a}{s^2+a^2}$, $s>0$.

8) $L[\cos at] = \frac{s}{s^2+a^2}$, $s>0$

Proof: (7) & (8)

By Euler's formula,

$e^{i\theta} = \cos\theta + i\sin\theta$

Replacing θ by at,

$e^{iat} = \cos(at) + i\sin(at)$

$L[e^{iat}] = L[\cos at] + iL[\sin at]$

$\frac{1}{s-ia} = L[\cos at] + iL[\sin at]$

$\frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} = L[\cos at] + iL[\sin at]$

$\frac{s+ia}{s^2-(ia)^2} = $ " $\Rightarrow \frac{s+ia}{s^2+a^2} = $ "

Equating real & imaginary parts

$L[\cos at] = \frac{s}{s^2+a^2}$

and

$L[\sin at] = \frac{a}{s^2+a^2}$.

9) $L[\sinh at] = \frac{a}{s^2-a^2}$

$s > |a|$

10) $L[\cosh at] = \frac{s}{s^2-a^2}$,

$s > |a|$

Hint: $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$\cosh at = \frac{e^{at} + e^{-at}}{2}$.

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Not all functions $f(t)$ are Laplace transformable.

For a function $f(t)$ to be Laplace transformable it must satisfy the following sufficient conditions

Sufficient Condition for Existence of Laplace Transforms.

Piecewise continuous.

A function $f(t)$ is said to be piecewise continuous in any interval $[a,b]$ if it is defined on that interval and such that the interval can be broken up into a finite number of sub-intervals in each of which $f(t)$ is continuous.

Function of Exponential Order:

A function $f(t)$ is said to be of exponential order

$$\lim_{t\to\infty} e^{-st} f(t) = 0.$$

Example: check if t^2 is of exponential order

$$\lim_{t\to\infty} e^{-st} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{st}} \quad \left(\frac{\infty}{\infty}\right) \text{form}$$

$$= \lim_{t\to\infty} \frac{2t}{s e^{st}} = \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = \lim_{t\to\infty} \frac{2}{s^2 e^{\infty}}$$

$$= \frac{2}{s^2(\infty)} = \frac{2}{s^2(1/0)} = 0 \ (\text{finite})$$

$\therefore$ t^2 is of exponential order.

Sufficient conditions for the existence of Laplace Transforms.

* $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a,b]$ where $a>0$.
* $f(t)$ should be of exponential order.