Residues:

If $z=a$ is an isolated singular point of $f(z)$, then $f(z)$ can be expanded in a Laurent's series about $z=a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $b_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{1-n}} dz$

The coefficient b_n of $\frac{1}{(z-a)^n}$ in the Laurent's series of $f(z)$ is called the Residue of $f(z)$ at $z=a$.

Methods:

(*) Residue at a simple pole

If $z=a$ is a simple pole or pole of order 1, then

$[\text{Res of } f(z)]_{z=a}$

$$= \lim_{z\to a} (z-a) f(z)$$

(*) Residue at a pole of order r is given by

$[\text{Res of } f(z)]_{z=a}$

$$= \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

Cauchy's Residue Theorem:

If $f(z)$ is analytic at all points inside and on a simple closed curve C, except at a finite number of poles $z_1, z_2, \ldots, z_n$ within C, then

$$\int_C f(z)\, dz$$

$= 2\pi i$ [sum of the residues of $f(z)$ at $z_1, z_2, \ldots z_n$]

i.e., $\int_C f(z) dz$

$$= 2\pi i \sum_{i=1}^{n} \{\text{Res } f(z)\}_{z=a_i}$$

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Problems:

1) Find the residue of $\frac{z+2}{(z-2)(z+1)^2}$

Solution:

Here $z=2$ is a simple pole and $z=-1$ is a pole of order 2.

(i) Residue at a simple pole $z=2$ is

$$Res(z=2) = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} (z-2) \frac{(z+2)}{(z-2)(z+1)^2}$$

$$= \lim_{z\to 2} \frac{z+2}{(z+1)^2}$$

$$= \frac{2+2}{(2+1)^2} = \frac{4}{9}$$

$\therefore Res(z=2) = \frac{4}{9}$

(ii) Residue at $z=-1$ a pole of order 2

$Res(z=-1) =$

$$\lim_{z\to -1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to -1} \frac{1}{1!} \frac{d}{dz}\left[(z+1)^2 \frac{(z+2)}{(z-2)(z+1)^2}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right]$$

$$= \lim_{z\to -1}\left[\frac{z-2-z-2}{(z-2)^2}\right]$$

$$= \lim_{z\to -1}\left[\frac{-4}{(z-2)^2}\right]$$

$$= \cancel{\lim_{z\to -1}}\left[\frac{-4}{(-1-2)^2}\right]$$

$$= \frac{-4}{(-3)^2} = \frac{-4}{9}$$

$\therefore$ Residue at $z=-1$ is $-\frac{4}{9}$

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2) Find the residue of

$$f(z) = \frac{1}{(z+1)(z-2)^2}$$

Ans: $\frac{1}{9}$ ($z=-1$, order 1)

$-\frac{1}{9}$ ($z=2$, order 2)

3) Find the residue of $f(z) = \frac{z}{z^2+1}$ about each singularity.

Soln:

$$f(z) = \frac{z}{z^2+1}$$

$$= \frac{z}{z^2-i^2}$$

$$= \frac{z}{\quad}$$

$z=-i$ and i are simple poles.

Now, $\text{Res}(z=-i)$

$$= \lim_{z\to -i} (z-(-i))\, f(z)$$

$$= \lim_{z\to -i} (z+i)\frac{z}{z^2+1}$$

$$= \lim_{z\to -i} (z+i)\frac{z}{(z+i)(z-i)}$$

$$= \lim_{z\to -i} \frac{z}{(z-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$\text{Res}(z=-i) = \frac{1}{2}$$

Now, $\text{Res}(z=i)$

$$= \lim_{z\to i} (z-i)\, f(z)$$

$$= \lim_{z\to i} (z-i)\frac{z}{(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{z}{z+i} = \frac{i}{i+i} = \frac{i}{2i}$$

$$= \frac{1}{2}$$

$$\text{Res}(z=i) = \frac{1}{2}$$

4) Find the residue of $f(z) = \frac{1}{(z^2+1)^2}$ about each singularity.

Solution:

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{[(z-i)(z+i)]^2}$$

$$= \frac{1}{(z-i)^2(z+i)^2}$$

Here $z=i,-i$ are the poles of order 2.

Now,

$$\text{Res}(z=i) = \lim_{z\to i} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-i)^2 f(z)\right]$$

$$= \lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\left[(z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \lim_{z\to i} \left[\frac{(z+i)^2(0) - i^2(z+i)(1)}{((z+i)^2)^2}\right]$$

$$= \lim_{z\to i} \left[\frac{0-2(z+i)}{(z+i)^4}\right]$$

$$= \lim_{z\to i} \left[\frac{-2(z+i)}{(z+i)^4}\right]$$

$$= \frac{-2(2i)}{(2i)^4} = \frac{-2}{(2i)^3}$$

$$= \frac{-2}{8i^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

$$= -\frac{i}{4}$$

$$\therefore \text{Res}(z=i) = -\frac{i}{4}$$

Now,

$$\text{Res}(z=-i) = \lim_{z\to -i} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-(-i))^2 f(z)\right]$$

$$= \lim_{z\to -i} \frac{1}{1!}\frac{d}{dz}\left[(z+i)^2 \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \lim_{z\to -i} \frac{d}{dz}\left[\frac{1}{(z-i)^2}\right]$$

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$$= \lim_{z \to -i} \left[\frac{(z-i)^2(0) - 1.2(z-i)}{((z-i)^2)^2} \right]$$

$$= \lim_{z \to -i} \left[\frac{0 - 2(z-i)}{(z-i)^4} \right]$$

$$= \lim_{z \to -i} \left[\frac{-2(z-i)}{(z-i)^4} \right]$$

$$= \lim_{z \to -i} \frac{-2}{(z-i)^3} = \frac{-2}{(-2i)^3}$$

$$= \frac{-2}{(-2)^3 (i)^3} = \frac{-2}{-8(-i)} = \frac{-1}{4i} \times \frac{i}{i}$$

$$= \frac{-i}{4i^2} = \frac{-i}{-4}$$

$$= \frac{i}{4} \quad \therefore Res(z=-i) = \frac{i}{4}$$

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5) Using Cauchy's Residue theorem, evaluate

$$\int_C \frac{4-3z}{z(z-1)(z-2)} dz \text{ where}$$

c is the circle $|z| = \frac{3}{2}$

Soln:

Let

$$f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

poles are at

$z=0$	$z=1$	$z=2$
simple pole	Simple pole	Simple pole

Circle $|z| = \frac{3}{2}$

$\lvert 0\rvert = 0 < \frac{3}{2}$	$\lvert 1\rvert = 1 < \frac{3}{2}$	$\lvert 2\rvert = 2 > 1$
The pole $z=0$ lies within C	The pole $z=1$ lies within C	The pole $z=2$ lies outside C

$$\therefore Res[(f(z))]_{z=0}$$

$$= \lim_{z \to 0} (z-0) f(z)$$

$$= \lim_{z \to 0} z \left(\frac{4-3z}{z(z-1)(z-2)} \right)$$

$$= \lim_{z \to 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4}{(-1)(-2)} = \frac{4}{2} = 2$$

$$\therefore Res(f(z))_{z=1}$$

$$= \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{1(1-2)} = \frac{1}{1(-1)} = -1$$

$\therefore$ By Cauchy Residue Theorem,

$$\int_C f(z)\, dz = 2\pi i [\text{Sum of all Residues}]$$

$$= 2\pi i [2 + (-1)]$$

$$= 2\pi i \;//$$

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6) Evaluate $\int_C \frac{z}{(z-1)^2(z+1)}\,dz$ i) $|z|=\frac{1}{2}$ ii) $|z|=2$

Soln: Let $f(z) = \frac{z}{(z-1)^2(z+1)}$

Poles are at

$z=1$ order 2	$z=-1$ order 1, simple pole

circle $|z|=\frac{1}{2}$

$|1| > \frac{1}{2}$ lies outside C

$|-1| = 1 > \frac{1}{2}$ lies outside C.

(ii) $|z|=2$ $|1|<2$ $|1|<2$ lies inside C,

$\therefore \int_C \frac{z}{(z-1)^4(z+1)}\,dz$ at $|z|=\frac{1}{2}$ is $=0$

(ii) $|z|=2$

$z=1$ order 2	$z=-1$ order 1
$1<2$	$\lvert -1 \rvert = 1 < 2$

$\therefore z=1,-1$ lies inside C.

$\therefore \boxed{\text{Res } f(z)}_{z=-1} = \lim_{z\to -1}(z+1)\frac{z}{(z-1)^2(z+1)}$

$= \lim_{z\to -1} \frac{z}{(z-1)^2}$

$= \frac{-1}{(-2)^2} = \frac{-1}{4}$

$\therefore$ $[\text{Res } f(z)]_{z=1} = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 \cdot \frac{z}{(z-1)^2(z+1)}\right]$

$= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[\frac{z}{z+1}\right]$

$= \lim_{z\to 1}\left(\frac{(z+1)(1) - z(1)}{(z+1)^2}\right)$

$= \lim_{z\to 1} \frac{z+1-z}{(z+1)^2} = \frac{1}{4}.$

By Cauchy's Residue Theorem,

$\int_C f(z)\,dz = 2\pi i\left(\sum(\text{Res})\right)$

$= 2\pi i\left(\frac{-1}{4}+\frac{1}{4}\right) = 0.$

7) H.W $f(z) = \int_C \frac{z-3}{(z+1)(z+2)}\,dz$

where C is the circle $|z|=3$

Ans: $2\pi i$ $(-4, 5)$

8) $f(z) = \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$

where C is a circle of $|z|=3$.

$z=1$ S.P

$z=2$ S.P

Res at $z=1$ $=1$

Res at $z=2 = 1$

Ans: $4\pi i$

9) Evalue $\int_C \frac{1}{(z^2+4)^2}\,dz$

where c is the circle $|z-i| = 2$.

Solution:

$f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z^2-(2i)^2)^2}$

$= \frac{1}{(z+2i)^2(z-2i)^2}$

$z+2i=0$	$z-2i=0$
$z=-2i$	$z=2i$

$z=\pm 2i$ are the poles of order 2.

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Given circle $|z-i|=2$

$z=+2i$ | $z=-2i$

$|2i-i|=|i|=1<2$ | $|-2i-i|$ $=|-3i|=9>2$

$z=+2i$ lies inside c

and $z=-2i$ lies outside c.

$\therefore \int_C f(z)\,dz = 2\pi i\,(\Sigma R)$

$[\text{Res } f(z)]_{z=2i}$

$$= \frac{1}{(2-1)!} \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \frac{1}{(z+2i)^2 (z-2i)^2}$$

$$= \lim_{z\to 2i}\left(\frac{1}{(z+2i)^2}\right)$$

$$= \lim_{z\to 2i} \frac{-2}{(z+2i)^3}$$

$$= \frac{-2}{(2i+2i)^3} = \frac{-2}{(4i)^3} = \frac{-2}{-64i}$$

$$= \frac{+1}{32i}$$

$\therefore$ By CRT

$$\int_C f(z)\,dz = 2\pi i\left[+\frac{1}{32i}+0\right]$$

$$= \frac{+\pi i}{16i} = +\frac{\pi}{16}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i[-1+0]$$

$$= -2\pi i$$ //.

10) Using Cauchy's Residue Theorem, Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}\,dz$, where C is the circle $|z-2|=\frac{1}{2}$

Solution:

$$f(z) = \frac{z}{(z-1)(z-2)^2}$$

$z-1=0$ | $z-2=0$

$z=1$ | $z=2$

$z=1$ is a simple pole

$z=2$ is a pole of order '2'

Gn circle $|z-2|=\frac{1}{2}$

$z=1$ | $z=2$

$|1-2|=|-1|$ | $|2-2|=|0|$

$=1>\frac{1}{2}$ | $=0<\frac{1}{2}$

$z=2$ lies inside the circle.

$z=1$ lies outside the circle,

$\therefore (\text{Res } f(z))_{z=1} = 0$

$[\text{Res } f(z)]_{z=2}$

$$= \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d}{dz}(z-2)^2 \frac{z}{(z-1)(z-2)^2}$$

$$= \lim_{z\to 2} \frac{d}{dz} \frac{z}{z-1} = \lim_{z\to 2} \left\{\frac{(z-1)-z}{(z-1)^2}\right\}$$

$$= \lim_{z\to 2} \frac{-1}{(z-1)^2} = \frac{-1}{1} = -1$$

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