UNIT-IV

Complex Integration.

Let $f(z)$ be a continuous function of the complex variable $z = x+iy$, along the arc C joining the point A and B, whose length is finite. Divide C into n parts by the points

$A = z_0, z_1, z_2, \ldots, z_{n-1}, z_n = B$

Let $\delta z_K = z_K - z_{K-1}$

and let α_K be any point in the arc $z_{K-1} z_K$.

Then the limit of the sum $\sum_{K=1}^{n} f(\alpha_K)\,\delta z_K$ as $n \to \infty$ in such a way that the length of every chord $\delta z_K \to 0$, is called the line integral $f(z)$ along C from A to B is denoted by $\int_C f(z)\,dz$.

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{K=0}^{n} f(z_k)\,\delta z_k$$

while each $\delta z_k \to 0$.

Cauchy's Integral Theorem

Statement:

If $f(z)$ is analytic and $f'(z)$ is continuous at all points inside and on a simple closed curve c, then

$$\int_C f(z)\,dz = 0$$

Cauchy's Integral formula (or) Cauchy's fundamental formula.

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz,$$

the integration around C being taken in the positive direction.

i.e., $\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$

Cauchy's Integral formula for derivatives

If $f(z)$ is analytic inside and on a simple closed curve C and $z=a$ is any interior point of the region R enclosed by C, then $f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} dz$,

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3} dz, \ldots$$

In general,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}},$$

the integration around C being taken in anti-clockwise direction.

i.e., $\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$

$$\int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

$\vdots$

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Note: If the pt lies outside the circle, $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} dz = 0$.

Problems:

1) Evaluate, using Cauchy's integral formula

$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$ on the circle (i) $|z|=4$ and

(ii) $|z|=1$

Solution:

Cauchy's Integral Formula is given by

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \text{--- (1)}$$

Given: $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$

on comparing with LHS of (1) we have,

$f(z) = z^2+5$

$a = 3$

$\therefore f(a) = f(3) = 3^2+5$

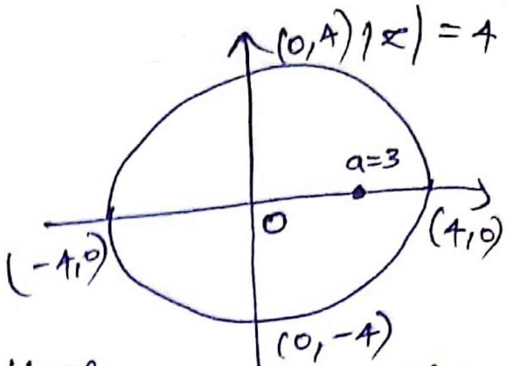
$= 9+5$

$f(3) = 14.$

$|z| = 4 \Rightarrow \sqrt{x^2+y^2} = 4$

$\Rightarrow x^2+y^2 = 4^2$

centre $C(0,0)$, radius -4



Here, $a=3$ lies within the circle $|z|=4$

$\therefore$ By (1)

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = \frac{1}{2\pi i}\cdot 2\pi i\,(14)$$

$= 14$

$$\boxed{\therefore \frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = 14}$$

(2) Evaluate $\int_C \frac{z\,dz}{z-2}$ where C is the circle $|z|=1$

Solution:

Cauchy's Integral Formula is given by

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \text{--- (1)}$$

Given: $\int_C \frac{z}{z-2}\,dz.$

on comparing with LHS of (1)

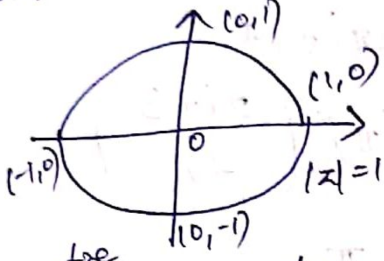
we have,

$f(z) = z$

and $a = 2$.

$\therefore f(a) = f(2) = 2$.

Given: Circle $|z| = 1$



centre $(0,0)$; radius 1

Here, $a = 2$ lies outside the circle C,

$$\therefore \int_C \frac{z}{z-2}\,dz = 0$$

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③ Evaluate $\int_C \frac{z^2-z+1}{z-1}\,dz$ where C is the circle (i) $|z| = 1$ & (ii) $|z| = \frac{1}{2}$, Using Cauchy's Integral Formula.

Solution:

Cauchy's Integral Formula is given by

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \quad \text{— ①}$$

Given: $\int_C \frac{z^2-z+1}{z-1}\,dz$

on comparing with LHS of ①

$f(z) = z^2 - z + 1$

$a = 1$

$\therefore f(a) = f(1)$

$= 1 - 1 + 1 = 1$

(i)

Given: Circle $|z| = 1$

unit circle

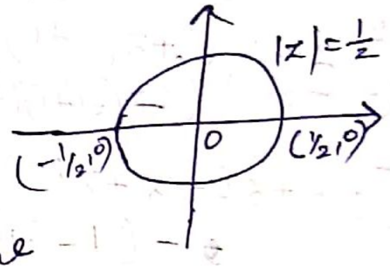
Here $a = 1$ lies on the circle.

$$\therefore \int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i(1)$$

$$= 2\pi i$$

(ii)

Given: Circle $|z| = \frac{1}{2}$ $\quad x^2+y^2 = \left(\frac{1}{2}\right)^2$



$(0,0)$

$r = \frac{1}{2}$

Here

$a = 1$ lies outside the circle.

$$\therefore \int_C \frac{z^2-z+1}{z-1}\,dz = 0.$$

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④ Using Cauchy's Integral formula, find the value of $\int_C \frac{z+4}{z^2+2z+5}\,dz$. where C is the circle $|z+1-i| = 2$.

Solution:

Cauchy's Integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Given: $\int_C \frac{z+4}{z^2+2z+5}\,dz$.

factorize.

$z^2+2z+5 \Rightarrow \frac{-2 \pm \sqrt{4-4 \cdot 5}}{2}$

$= \frac{-2 \pm \sqrt{4-20}}{2}$

$= \frac{-2 \pm 4\sqrt{-1}}{2}$

$= -1 \pm 2i$

$\therefore z = (-1+2i) \quad z = (-1-2i)$

$z^2+2z+5 = (z-(-1+2i))(z-(-1-2i))$

$\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}$

$= \int_C \frac{z+4}{[z-(-1+2i)][z+(1+2i)]}$

$= \int_C \frac{(z+4)/[z+(1+2i)]}{[z-(-1+2i)]}$

Now comparing with LHS of ①

$f(z) = \frac{z+4}{z+1+2i}$

and $a = -1+2i$

$f(-1+2i) = f(a) = \frac{-1+2i+4}{-1+2i+1+2i}$

$f(a) = \frac{2i+3}{4i}$

④ Include

$\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz$

$= 2\pi i f(a)$

$= \cancel{2}\pi \cancel{i} \left(\frac{2i+3}{\cancel{4}_2 \cancel{i}} \right)$

$= \frac{\pi}{2}(2i+3)$

$\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz = \frac{\pi}{2}(2i+3)$

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⑤ Evaluate using Cauchy's integral formula $\int_C \frac{z}{(z-1)(z-2)}\,dz$, where C is the circle $|z-2| = \frac{1}{2}$

Solution:

By Cauchy's integral formula,

$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Given: $\int_C \frac{z}{(z-1)(z-2)}\,dz$

Consider $\dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$1 = A(z-2) + B(z-1)$

put $z=2 \Rightarrow B = 1$

put $z=1 \Rightarrow A = -1$

$\therefore \dfrac{1}{(z-1)(z-2)} = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

$\displaystyle\int_C \frac{z}{(z-1)(z-2)} dz = \int_C \left(\frac{-z}{z-1} + \frac{z}{z-2}\right) dz$

$\displaystyle = \int_C \frac{-z}{z-1} dz + \int_C \frac{z}{z-2} dz$

$= I_1 + I_2$

$\displaystyle I_1 = \int_C \frac{-z}{z-1} dz$

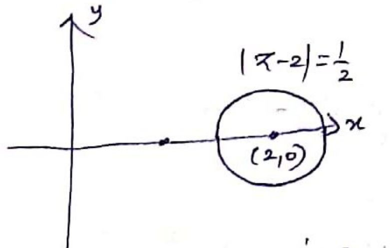
Given circle $|z-2| = \frac{1}{2}$

$|x+iy-2| = \frac{1}{2}$

$|x-2+iy| = \frac{1}{2}$

$(x-2)^2 + y^2 = \frac{1}{2^2}$

centre $(2,0)$ and radius $\frac{1}{2}$



on comparing, $f(z) = -z$

$a = 1$

$f(a) = f(1) = -1$

$a=1$ lies outside the circle

$\displaystyle \therefore \int_C \frac{-z}{z-1} dz = 0$

$\displaystyle I_2 = \int_C \frac{z}{z-2} dz$

$f(z) = z$ & $a = 2$

$f(a) = 2$

$a=2$ lies also inside the circle.

$\displaystyle \therefore \int_C \frac{z}{z-2} dz = 2\pi i(2)$

$= 4\pi i$

$\displaystyle \therefore \int_C \frac{z\,dz}{(z-1)(z-2)} = 0 + 4\pi i$

$= 4\pi i$

(6) Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$, where C is the circle $|z|=3$.

Solution:

By Cauchy Integral formula, $\displaystyle\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$.

Given: $\displaystyle\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$.

Consider $\dfrac{e^{2z}}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$\Rightarrow e^{2z} = A(z-2) + B(z-1)$

put $z=2$,

$e^{2(2)} = A(0) + B(1)$

$\Rightarrow \boxed{B = e^4}$

put $z=1$,

$e^{2(1)} = A(-1) + B(0)$

$\Rightarrow \boxed{-A = e^2}$

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)} = \int_C \frac{-e^2}{(z-1)} + \frac{e^4}{(z-2)} dz$$

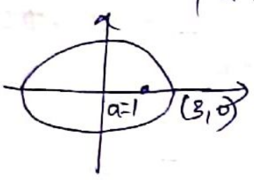
$$I_1 = \int_C \frac{-e^2}{z-1}$$

$f(z) = -e^2$ and $a = 1$

$f(a) = f(1) = -e^2$

Given circle $|z| = 3$

$|x+iy| = 3 \Rightarrow x^2+y^2 = 3^2$



$a = 1$ lies inside the circle.

$$\therefore \int_C \frac{-e^2}{z-1} = 2\pi i f(1)$$
$$= 2\pi i(e^2)$$
$$= -e^2(2\pi i)$$

$$I_2 = \int_C \frac{e^4}{z-2}$$

$f(z) = e^4$ and $a = 2$

for the, $f(a) = f(2) = e^4$

Given circle $|z| = 3$,

$a = 2$ lies inside the circle.

$$\therefore \int_C \frac{e^4}{z-2} = 2\pi i f(2)$$
$$= 2\pi i(e^4)$$
$$= e^4(2\pi i).$$

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)} = -e^2(2\pi i) + e^4(2\pi i)$$
$$= 2\pi i\left(e^4 - e^2\right) //$$

7) Evaluate

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$$

where C is $|z| = 3$.

Solution:

~~Using~~ Cauchy's Integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i \, f(a).$$

Given: $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$

Consider

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(using partial fraction)

$$\therefore \int_C {}^{\prime\prime} dz = \int_C \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

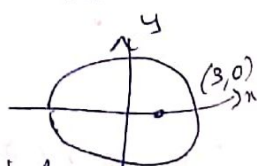
$$\int_C \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz$$

$f(z) = -(\sin \pi z^2 + \cos \pi z^2)$

$a = 1$

$f(a) = f(1) = -\sin \pi - \cos \pi$

$a = 1$, $|z| = 3$



point lies inside the circle

$$\therefore \int_C {}^{\prime\prime} dz = 2\pi i f(a)$$
$$= 2\pi i(-\sin \pi - \cos \pi).$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$

$a = 2$

$f(a) = \sin 4\pi + \cos 4\pi$

$a=2$, lies inside the circle $|z|=3$

$$\int_C \text{''} \, dz = 2\pi i (\sin 4\pi + \cos 4\pi).$$

W.K.T $\cos \pi = -1$

$\cos n\pi = 1$ if n is even

$= -1$ if n is odd.

$$\therefore \int_C \text{''} \, dz = 2\pi i(-\sin\pi - \cos\pi) + 2\pi i (\sin 4\pi + \cos 4\pi)$$

$$= 2\pi i (0 - \cancel{\cos}(-1)) + 2\pi i (0+1)$$

$$= +2\pi i + 2\pi i = 4\pi i$$

$$\int_C \text{''} \, dz = 4\pi i$$

H.W

1) $\int_C \frac{3z^2+z}{z^2-1} dz$, $|z-1|=1$

2) $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, $|z|=3$

3) $\int_C \frac{e^{-z}}{z+1} dz$ (i) $|z|=2$ (ii) $|z|=\frac{1}{2}$

4) $\int_C \frac{z+1}{z^2+2z+4} dz$ $|z+1+i|=2$

5) $\int_C \frac{e^{2z}}{z^2+1} dz$, $|z|=\frac{1}{2}$

Hint $z^2+1 = (z+i)(z-i)$

6) $\int_C \frac{\cos \pi z}{z-1} dz$, $|z|=2$ Ans: $-2\pi i$

1) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z|=2$.

Solution: (ii) $|z-1|=2$

Cauchy's Integral formula for derivatives

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Given: $\int_C \frac{e^{2z}}{(z+1)^4} dz$

$f(z) = e^{2z}$ | $n+1=4$, $n=3$

$a = -1$

Given circle: $|z|=2$

centre $(0,0)$ & $R(2)$

$a=-1$ lies inside the circle

$n! = 3! = 3\times 2\times 1 = 6$

$f^{(3)}(a) = f'''(a)$

$f(z) = e^{2z}$

$f'(z) = e^{2z}\cdot 2$

$f''(z) = 2\left[e^{2z}\cdot 2\right]$

$f'''(z) = 4\left[e^{2z}\cdot 2\right]$

$f'''(a) = f'''(-1) = 4\left[e^{2(-1)}\cdot 2\right]$

$= 8e^{-2}$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{6}(8e^{-2})$$

$$= \frac{8\pi i}{3} e^{-2}$$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3} e^{-2}$$

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2) Evaluate $\int_C \frac{z^3-z}{(z-2)^3}\,dz$, where C is the circle $|z|=3$.

3) $\int \frac{e^{-2z}\,dz}{(z+1)^3}$, $|z|=2$

4) $\int_C \frac{z^2+1}{z^2-1}\,dz$, $|z-1|=1$

2) Evaluate $\int_C \frac{z^3-z}{(z-2)^3}\,dz$, where C is the circle $|z|=3$.

Solution:

Cauchy's Integral formula for derivatives is given by $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$

Gn: $\int_C \frac{z^3-z}{(z-2)^3}\,dz$.

Gn: circle $|z|=3$, $c(0,0)$, $r(3)$

$a=2$ lies inside the circle

$f(z) = z^3 - z$

$f'(z) = 3z^2 - 1$

$f''(z) = 6z$

$\therefore \int_C \frac{z^3-z}{(z-2)^{2+1}}$

$= \frac{2\pi i}{2!} f''(2)$

$= \frac{2\pi i}{2}\, 6(2) = 12\pi i$

3) Evaluate $\int_C \frac{dz}{z^2 e^z}$, $|z|=1$

Soln:

$\int_C \frac{e^{-z}}{z^2}\,dz = \int_C \frac{e^{-z}}{(z-0)^2}\,dz$

$a=0$ | $f(z) = e^{-z}$

$f'(z) = -e^{-z}$

Given: $|z|=1$

$c(0,0)$ $r=1$

$a=0$ lies inside the circle.

$\therefore \int_C \frac{e^{-z}}{(z-0)^2}\,dz = \frac{2\pi i}{1!} f'(a)$

$= \frac{2\pi i}{1}(-e^{-0})$

$= 2\pi i(-1)$

$= -2\pi i$

5) $\int_C \frac{dz}{(z-3)^2}$ where c is the circle $|z|=1$

Ans $=0$

6) $\int_C \frac{e^z\,dz}{z+1}$, $\left|z+\frac{1}{2}\right|=1$ Ans: $4\pi i$

7) $\int_C \frac{dz}{z^2-2z}$, $|z|=1$

Ans: $-\pi i$