Gauss Divergence Theorem Solution

Statement:

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

i.e., $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$

where $\hat{n}$ is the unit vector in the positive (outward drawn) normal to S.

Problem-1

1) Verify the Gauss Divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0$; $x=1$; $y=0$; $y=1$; $z=0$; $z=1$.

Note: 1) The significance of the Gauss's divergence theorem lies in the fact that a surface integral may be expressed as a volume integral and vice versa.

Solution

Given:

$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$.

RHS of G.D.T.

$\iiint_V \nabla\cdot\vec{F}\,dv$.

$\nabla\cdot\vec{F}$

$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$

$= \frac{\partial}{\partial x}4xz - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}(yz)$

$= 4z - 2y + y$

$\boxed{\nabla\cdot\vec{F} = 4z - y}$

Since V is volume,

$dv = dx\,dy\,dz$.

Given x, y, z varies from 0 to 1

$$\therefore \; RHS = \iiint_V \nabla \cdot \vec{F} \, dv$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^1 \left\{ \int_0^1 (4z - y) \, dx \right\} dy \, dz$$

$$= \int_0^1 \int_0^1 \left[4xz - yx \right]_0^1 dy \, dz$$

$$= \int_0^1 \int_0^1 (4z - y) \, dy \, dz$$

$$= \int_0^1 \left[4yz - \frac{y^2}{2} \right]_0^1 dz$$

$$= \int_0^1 4\cancel{y}z - \frac{1}{2} \, dz$$

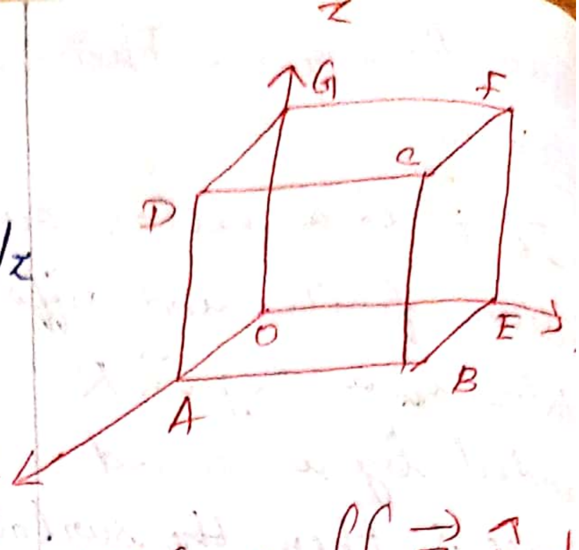
$$= \left[4\frac{z^2}{2} - \frac{1}{2} z \right]_0^1$$

$$= \left[2z^2 - \frac{z}{2} \right]_0^1$$

$$= \left[2 - \frac{1}{2} \right] - 0$$

$$= \frac{4-1}{2} = \frac{3}{2}.$$

$$\therefore \; RHS = \frac{3}{2}. \quad \text{(1)}$$



$$\underline{LHS} = \iint_S \vec{F} \cdot \hat{n} \, d$$

Since there are six surfaces of the given cube (faces − 6)

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

S_1 = ABCD

S_2 = OEFG

S_3 = BEFC

S_4 = AOGD

S_5 = CDGF

S_6 = OABE

Face	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$	$I_1 = \iint_{S_1} \vec{F}\cdot\hat{n}\, ds$
S_1 $x=1$	$\vec{i}$	$dy\,dz$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot\vec{i}$ $= 4xz$ $= 4z$	$I_1 = \int_0^1\int_0^1 4z\,dy\,dz$ $= \int_0^1 4z[y]_0^1\,dz$ $= \int_0^1 4z(1-0)\,dz$ $= 4\left[\frac{z^2}{2}\right]_0^1$ $= 4\left(\frac{1}{2}-0\right)$ $= 4\left(\frac{1}{2}\right) = 2$ $\boxed{I_1 = 2}$
S_2 $x=0$	$-\vec{i}$	$dy\,dz$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(-\vec{i})$ $= 4xz(-1)$ $= -4xz$ $= 0 \quad (\because x=0$	$I_2 = \int_0^1\int_0^1 0\,dy\,dz$ $\boxed{I_2 = 0}$
S_3 $y=1$	$\vec{j}$	$dx\,dz$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(\vec{j})$ $= (-y^2)(1)$ $= -y^2$ $= -1$	$I_3 = \int_0^1\int_0^1 (-1)\,dx\,dz$ $= -\int_0^1 [x]_0^1\,dz$ $= -\int_0^1 (1-0)\,dz$ $= -[z]_0^1$ $= -1$ $\boxed{I_3 = -1}$

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face	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$	$I = \iint_S \vec{F}\cdot\hat{n}\, ds$
S_4 $y=0$	$-\vec{j}$	$dx\,dz$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(-\vec{j})$ $=(-y^2)(-1)$ $=y^2$ $=0$	$I_4=\int_0^1\int_0^1 0\, dx\, dz$ $I_4=0$
S_5 $z=1$	$\vec{k}$	$dx\,dy$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(\vec{k})$ $=yz$ $=y$	$I_5=\int_0^1\int_0^1 y\, dx\, dy$ $=\int_0^1 [yx]_0^1\, dy$ $=\int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1$ $\boxed{I_5=\frac{1}{2}}$
S_6 $z=0$	$-\vec{k}$	$dx\,dy$	$(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(-\vec{k})$ $=(yz)(-1)$ $=0$	$I_6=\int_0^1\int_0^1 0\, dx\, dy$ $\boxed{I_6=0}$

$\therefore$ LHS $= I_1+I_2+I_3+I_4+I_5+I_6$.

$= 2+0+(-1)+0+\left(\frac{1}{2}\right)+0$

$= 1+\frac{1}{2}$

LHS $= \frac{3}{2}$ —— (2)

From (1) & (2)

LHS = RHS.

Hence Gauss-Divergence Theorem is verified.

2) Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$, taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Soln:

Given:

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}.$$

RHS of GDT

$$\iiint_V \nabla \cdot \vec{F}\, dv$$

~~dv = dx dy dz.~~

$\nabla \cdot \vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\cdot\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z.$$

$$= 2(x+y+z)$$

Since v is volume,

$dv = dx\,dy\,dz.$

Given: x varies from 0 to a

y varies from 0 to b

z varies from 0 to c

$$RHS = \int_0^c \int_0^b \int_0^a 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^c \int_0^b \left[\frac{x^2}{2} + yx + zx\right]_0^a dy\, dz$$

$$= 2\int_0^c \int_0^b \left[\frac{a^2}{2} + ya + za\right] dy\, dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + \frac{y^2}{2}a + zay\right]_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}b + \frac{b^2 a}{2} + zab\right] dz$$

$$= 2\left[\frac{a^2 b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2 b}{2}c + \frac{ab^2}{2}c + \frac{abc^2}{2}\right]$$

$$= \left[a^2bc + ab^2c + abc^2\right]$$

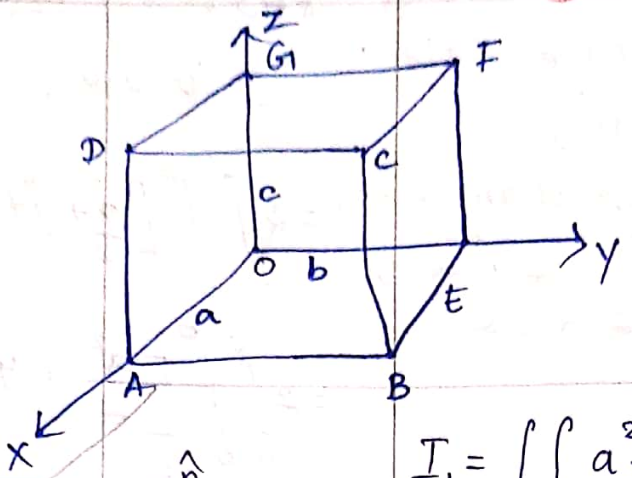
$$= abc\left[a+b+c\right] \text{ cu units}$$

$\therefore$ RHS $= abc[a+b+c]$ cu. units. (1)

LHS

$$\iint_S \vec{F}\cdot\hat{n}\, ds.$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

face	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$	$I=\iint_S \vec{F}\cdot\hat{n}\,ds$
$S_1 = ABCD$ $S_2 = OEFG$ $S_3 = BEFC$ $S_4 = OADG$ $S_5 = GDCF$ $S_6 = OABE$				
S_1 $x=a$	$\vec{i}$	$dydz$	$\vec{F}\cdot\hat{n}$ $\vec{F}=(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\cdot(\vec{i})$ $=x^2-yz$ $=a^2-yz.$	$I_1=\iint_{S_1} a^2-yz\,dy\,dz$ $=\int_0^c\int_0^b (a^2-yz)\,dy\,dz$ $=\int_0^c\left[a^2y-\frac{y^2z}{2}\right]_0^b dz$ $=\int_0^c\left[a^2b-\frac{b^2z}{2}\right]dz$ $=\left[a^2bz-\frac{b^2}{2}\frac{z^2}{2}\right]_0^c$ $=\left[a^2bc-\frac{b^2}{2}\frac{c^2}{2}\right]$ $=a^2bc-\frac{b^2c^2}{4}$
S_2 $x=0$	$-\vec{i}$	$dydz$	$\vec{F}\cdot\hat{n}$ $=-(x^2-yz)$ $=+yz$	$I_2=\iint_{S_2} yz\,dy\,dz$ $=\int_0^c\int_0^b yz\,dy\,dz=\int_0 z\left[\frac{y^2}{2}\right.$ $=\frac{b^2}{2}\int_0^c z\,dz=\frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$ $=\frac{b^2e^2}{4}$

Face	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$	$I = \iint_S \vec{F}\cdot\hat{n}\, ds$
S_3 $y=b$	$\vec{j}$	$dx\,dz$	$\vec{F}\cdot\hat{n}$ $= y^2 - zx$ $= (b^2 - zx)$	$I_3 = \iint_{S_3} (b^2 - zx)\,dx\,dz$ $= \int_0^c \int_0^a (b^2 - zx)\,dx\,dz$ $= \int_0^c \left(b^2x - z\frac{x^2}{2}\right)_0^a dz$ $= \int_0^c \left[b^2a - z\frac{a^2}{2}\right] dz$ $= \left[b^2a\,z - \frac{z^2}{2}\frac{a^2}{2}\right]_0^c$ $= \left[b^2ac - \frac{z^2a^2}{4}\right]$ $I_3 = ab^2c - \frac{a^2z^2}{4}.$
S_4 $y=0$	$-\vec{j}$	$dx\,dz$	$\vec{F}\cdot\hat{n}$ $= -(y^2 - zx)$ $= 0 + zx$	$I_4 = \iint_{S_4} zx\,dx\,dz$ $= \int_0^c \int_0^a zx\,dx\,dz$ $= \int_0^c z\cdot\left[\frac{x^2}{2}\right]_0^a dz$ $= \int_0^c z\frac{a^2}{2}\,dz$ $= \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^c$ $= \frac{a^2}{2}\left[\frac{c^2}{2}\right] = \frac{a^2c^2}{4}$

face	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$	$I = \iint_S \vec{F}\cdot\hat{n}\, ds$
S_5 $z=c$	$\vec{k}$	$dx\,dy$	$\vec{F}\cdot\hat{n}$ $= z^2 - xy$ $= c^2 - xy$	$I_5 = \iint_S c^2 - xy\, dx\, dy$ $= \int_0^b \int_0^a c^2 - xy\, dx\, dy$ $= \int_0^b \left[c^2 x - \frac{x^2}{2} y\right]_0^a dy$ $= \int_0^b c^2 a - \frac{a^2}{2} y\, dy$ $= \left[c^2 a y - \frac{a^2}{2}\frac{y^2}{2}\right]_0^b$ $= c^2 ab - \frac{a^2}{2}\frac{b^2}{2}$ $I_5 = abc^2 - \frac{a^2 b^2}{2}$
S_6 $z=0$	$-\vec{k}$	$dx\,dy$	$\vec{F}\cdot\hat{n}$ $= -(z^2 - xy)$ $= +xy$	$I_6 = \iint_S xy\, dx\, dy$ $= \int_0^b \int_0^a xy\, dx\, dy$ $= \int_0^b \left[\frac{x^2}{2} y\right]_0^a dy$ $= \int_0^b \frac{a^2}{2} y\, dy$ $= \frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^b$ $= \frac{a^2}{2}\frac{b^2}{2}$ $= \frac{a^2 b^2}{4}$

Now,

$$LHS = I = \iint_S \vec{F} \cdot \hat{n} \, ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

$$= \left(a^2bc - \frac{b^2c^2}{4}\right) + \frac{b^2c^2}{4}$$

$$+ \left(ab^2c - \frac{a^2c^2}{4}\right) + \frac{a^2c^2}{4}$$

$$+ \left(abc^2 - \frac{a^2b^2}{2}\right) + \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

$$LHS = abc(a+b+c) \quad \text{---} (2)$$

Hence from (1) & (2), Gauss Divergence theorem is verified.

HW

1) Verify divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over the cube bounded by $x=0$; $x=a$; $y=0$; $y=a$; $z=0$; $z=a$.