Harmonic functions :

An expression of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called the Laplace equation in two dimension.

Any function having continuous second order partial derivatives which satisfies the Laplace equation is called harmonic function.

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called Conjugate harmonic functions.

Note:

Both real and imaginary parts of an analytic function are harmonic. But the converse need not be true.

(1) Give an example such that u and v are harmonic but $u+iv$ is not analytic.

Soln:

Let $w = \bar{z}$

$u+iv = x-iy$

$\Rightarrow u = x \; ; \; v = -y$

$\frac{\partial u}{\partial x} = 1, \; \frac{\partial u}{\partial y} = 0 \; ; \; \frac{\partial v}{\partial x} = 0, \; \frac{\partial v}{\partial y} = -1$

$\frac{\partial^2 u}{\partial x^2} = 0, \; \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} = 0, \; \frac{\partial^2 v}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

$\Rightarrow u$ and v are harmonic.

But $u_x \neq v_y$ and $u_y = -v_x$

$\therefore f(z) = u+iv$ is not analytic.

(2) Prove that $u = e^x \cos y$ is a harmonic function.

Soln:

Let $u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y \; ; \; \frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \; ; \; \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore u$ is a harmonic function.

(3) Prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ are harmonic but $u+iv$ is not a regular function.

Soln:

Let $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$\Rightarrow u$ is a harmonic function.

Let $v = \frac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = -\frac{[(x^2+y^2)0 - y(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2 2y - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = -\frac{[(x^2+y^2) - y.2y]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2 2y - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)2y - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^3}{(x^2+y^2)^3} = 0.$$

$\Rightarrow$ v is a harmonic function.

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$f(z) = u+iv$ is <u>not analytic</u> (or) <u>not regular</u> function.

Construction of Conjugate harmonic fns :

Method 1 :

Suppose u is given, then

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$$, where c is a constant.

Method 2 :

Suppose v is given, then

$$u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy\right) + c$$, where c is a constant.

1. Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic Conjugate.

Soln :

Let $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

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$$u_y = \frac{1}{2}\,\frac{1}{x^2+y^2}\,2y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore\ u_{xx} + u_{yy} = 0$.

u satisfies Laplace equation

u is harmonic.

$$v = \int \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \right) + c$$

$$= \int \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \right) + c$$

$$= \int \frac{x\,dy - y\,dx}{x^2+y^2} + c$$

$$= \int \frac{x\,dy - y\,dx}{x^2(1+y^2/x^2)}$$

$$= \int \frac{d(y/x)}{1+y^2/x^2} + c$$

$$v = \tan^{-1}(y/x) + c$$

(2) Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic find the conjugate harmonic fn

Soln:

Let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$; $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$; $u_{yy} = -6x - 6$

$= -(6x+6)$

$u_{xx} + u_{yy} = 0$

$\therefore$ u satisfies Laplace equation.

$\Rightarrow$ u is harmonic

Now $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$

$= \int (6xy + 6y)\, dx + (3x^2 - 3y^2 + 6x)\, dy + c$

$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy + c$

$= \frac{1}{2}\left[6x^2y + 12xy + 6x^2y - 2y^3 + 12xy + 2c\right]$

$v = 6x^2y + 12xy - y^3 + c$

③ ST $u = \cos x \cosh y$ is harmonic & hence find its harmonic conjugate

Soln:

$u = \cos x \cosh y$

$u_x = -\sin x \cosh y$; $u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y$; $u_{yy} = \cos x \cosh y$

$u_{xx} + u_{yy} = 0$.

$\Rightarrow$ u satisfies Laplace eqn

$\Rightarrow$ u is harmonic.

Now $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$

$= \int (-\cos x \sinh y\, dx) + (-\sin x \cosh y)\, dy + c$

$= -\sin x \sinh y - \sin x \sinh y + c$

$v = -2 \sin x \sinh y + c$