Divergence and Curl

Defn:

1) Let $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be a vector valued function. The divergence of f denoted by $\nabla.f$ (or) $\mathrm{Div}\,\vec{f}$ is defined by

$$\nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \sum \vec{i}.\frac{\partial f}{\partial x}$$

2) The curl of $\vec{f}$ denoted by $\nabla \times \vec{f}$ or curl $\vec{f}$ is defined to be $\sum \vec{i} \times \frac{\partial f}{\partial x}$.

$$\text{curl } f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

div of $f \rightarrow$ Scalar valued fun

curl of $f \rightarrow$ vector valued fun.

Solenoidal Vector:

A Vector $\vec{f}$ is called solenoidal if div $\vec{f} = 0$.

Irrotational Vector:

A Vector $\vec{f}$ is called irrotational if curl $\vec{f} = 0$.

Note:

$\nabla.\vec{F}$ - Scalar quantity

$\nabla \times \vec{F}$ - Vector quantity

1) If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ then find $\nabla . \vec{F}$ and $\nabla \times \vec{F}$.

Solution:

$$\text{Div}\vec{F} = \nabla . \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

since $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\text{curl}\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= \vec{0} = 0$$

H.W:

2) Find the divergence and curl of a vector point function $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$.

Soln: Let $\vec{F} =$ "

$$\text{Div}\vec{F} = \nabla . \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$\therefore \text{Div}\vec{F} = y^2 + 2x^2z - 6yz$$

$$\text{curl}\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \vec{i}\left[-3z^2 - 2x^2y\right] - \vec{j}\left[0-0\right] + \vec{k}\left[4xyz - 2xy\right]$$

$$\text{curl}\vec{F} = \vec{i}(-3z^2 - 2x^2y) + \vec{k}(4xyz - 2xy)$$

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3) Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ of the vector point function $\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$ at the point $(1, -1, 1)$

Soln:

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2z + 8yz^3$$

$$\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = z^3 - 2x^2z + 8z^3y$$

$$(\nabla \cdot \vec{F})_{(1,-1,1)} = 1^3 - 2(1)(1) + 8(1)(-1)$$

$$= 1 - 2 - 8 = -9$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

1) Prove that

curl of $(\nabla\phi) = 0$ (or)

$\nabla \times \nabla\phi = 0$.

Soln:

$$\text{curl } \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y \partial z} - \frac{\partial^2\phi}{\partial z \partial y}\right)$$

$$-\vec{j}\left(\frac{\partial^2\phi}{\partial x \partial z} - \frac{\partial^2\phi}{\partial z \partial x}\right)$$

$$+\vec{k}\left(\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial y \partial x}\right)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0.$$

$\therefore \nabla \times \nabla\phi = 0$.

(or)

curl $\nabla\phi = 0$