Triple Integration in Cartesian Co-ordinates

Triple Integration of a function defined over a region $\iiint\limits_R f(x, y, z)\, dx\, dy\, dz$.

1) $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$.

$= \int_0^3 \int_0^2 \left[\int_0^1 (x+y+z)\, dz \right] dy\, dx$.

$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_{z=0}^{z=1} dy\, dx$.

$= \int_0^3 \int_0^2 \left[x + y + \frac{1}{2} \right] dy\, dx$.

$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{1}{2}y \right]_0^2 dx$

$= \int_0^3 \left(2x + \frac{4}{2} + \frac{2}{2}\right) - (0)\, dx$.

$= \int_0^3 \left(2x + \frac{4}{2} + 1\right) dx$

$= \left[\frac{2x^2}{2} + 2x + x \right]_0^3$

$= \left[3^2 + 2(3) + 3 \right] = 9 + 6 + 3 = 18$

$\boxed{\therefore I = 18}$

2) Evaluate $\int_0^1\int_0^2\int_0^3 xyz\, dx\, dy\, dz$

Soln:

$$I = \int_0^1\int_0^2\int_0^3 xyz\, dx\, dy\, dz.$$

$$= \int_0^1\int_0^2 \left[\frac{x^2}{2}\right]_0^3 yz\, dy\, dz$$

$$= \int_0^1\int_0^2 \left[\frac{3^2}{2}\right] yz\, dy\, dz.$$

$$= \frac{9}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 z\, dz.$$

$$= \frac{9}{2}\left[\frac{4}{2}\right]\int_0^1 z\, dz$$

$$= \frac{36}{4}\left[\frac{z^2}{2}\right]_0^1 = 9\left[\frac{1}{2}\right] = \frac{9}{2}$$

$$\therefore \boxed{I = \frac{9}{2}}$$

3) Evaluate $\int_0^a\int_0^b\int_0^c e^{x+y+z}\, dx\, dy\, dz$

Soln:

$$I = \int_0^a\int_0^b\int_0^c e^{x+y+z}\, dx\, dy\, dz.$$

$$= \int_0^a\int_0^b\int_0^c e^y e^z e^x\, dx\, dy\, dz$$

$$= \int_0^a\int_0^b e^y e^z \left[e^x\right]_0^c dy\, dz.$$

$$= \int_0^a\int_0^b e^y e^z \left[e^c - e^0\right) dy\, dz.$$

$$= (e^c-1)\int_0^a e^{z}\left(\int_0^b e^y\, dy\right) dz$$

$$= (e^c-1)\int_0^a e^z \left[e^y\right]_0^b dz$$

$$= (e^c-1)\int_0^a e^z (e^b - e^0)\, dz$$

$$= (e^c-1)\int_0^a e^z (e^b-1)\, dz$$

$$= (e^c-1)(e^b-1)\int_0^a e^z\, dz$$

$$= (e^c-1)(e^b-1)\left[e^z\right]_0^a$$

$$= (e^c-1)(e^b-1)(e^a - e^0)$$

$$= (e^c-1)(e^b-1)(e^a-1).$$

$$\therefore I = (e^c-1)(e^b-1)(e^a-1)$$

4) Evaluate $\int_0^1\int_0^y\int_0^{x+y} dx\,dy\,dz$

lution:

et $I = \int_0^1\int_0^y\int_0^{x+y} dx\,dy\,dz$

uiven integration is not in ie correct form, so the orrect form is

$$I = \int_0^1\int_0^y\int_0^{x+y} dz\,dx\,dy.$$

$$= \int_0^1\int_0^y [z]_0^{x+y} dx\,dy.$$

$$= \int_0^1\int_0^y (x+y)-0)\,dx\,dy$$

$$= \int_0^1\int_0^y [x+y]\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2}+xy\right]_0^y dy$$

$$= \int_0^1 \left(\left[\frac{y^2}{2}+y^2\right)-0\right) dy$$

$$= \int_0^1 \frac{y^2}{2}+y^2\,dy$$

$$= \left[\frac{y^3}{2\times 3}+\frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1^3}{6}+\frac{1^3}{3}\right]-[0]$$

$$= \left[\frac{1+2}{6}\right] = \left[\frac{3}{6}\right] = \frac{1}{2}$$

$$\therefore \boxed{I = \frac{1}{2}}$$

5) $\int_0^a\int_0^b\int_0^c dz\,dy\,dx$

Let $I = \int_0^a\int_0^b\int_0^c dz\,dy\,dx.$

$$= \int_0^a\int_0^b [z]_0^c\,dy\,dx$$

$$= \int_0^a\int_0^b (c-0)\,dy\,dx$$

$$= \int_0^a\int_0^b c\,dy\,dx$$

$$= \int_0^a c\,[y]_0^b\,dx$$

$$= \int_0^a c\,[b-0]\,dx.$$

$$= \int_0^a cb\,dx$$

$$= cb\,[x]_0^a$$

$$= cb\,[a-0] = cba$$

$$\boxed{\therefore I = abc}$$

6) $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx.$

Solution:

Let $I = \int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx$

$$= \int_0^2\int_1^3 xy^2\left[\frac{z^2}{2}\right]_1^2 dy\,dx.$$

$$= \int_0^2\int_1^3 xy^2\left[\frac{4}{2}-\frac{1}{2}\right] dy\,dx$$

$$= \int_0^2\int_1^3 xy^2\left[2-\frac{1}{2}\right] dy\,dx.$$

$$= \int_0^2 \int_1^3 xy^2 \frac{3}{2}\, dy\, dx$$

$$= \frac{3}{2} \int_0^2 \left[x \frac{y^3}{3} \right]_1^3 dx$$

$$= \frac{3}{2} \int_0^2 x \left[\frac{27}{3} - \frac{1}{3} \right] dx$$

$$= \frac{3}{2} \int_0^2 x \left[\frac{26}{3} \right] dx$$

$$= \frac{26}{2} \int_0^2 x\, dx$$

$$= 13 \left[\frac{x^2}{2} \right]_0^2$$

$$= 13 \left[\frac{4}{2} - \frac{0}{2} \right] = 13 \left[2 - \cancel{\tfrac{0}{2}} \right]$$

$$\cancel{= 13 \left[\frac{3}{2} \right] = \frac{39}{2}} \quad \boxed{I = 26}$$

$$\therefore \boxed{\cancel{I = \frac{39}{2}}}$$

7) Evaluate $\int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx$.

Solution:

Let $I = \int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx$

$$= \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right]_0^c dy\, dx.$$

$$= \int_0^a \int_0^b \left[xc + yc + \frac{c^2}{2} \right] - [0]\, dy.\, dx$$

$$= \int_0^a \int_0^b \left[cx + cy + \frac{c^2}{2} \right] dy\, dx$$

$$= \int_0^a \int_0^b \left(cx + cy + \frac{c^2}{2} \right) dy\, dx$$

$$= \int_0^a \left[cxy + \frac{cy^2}{2} + \frac{c^2 y}{2} \right]_{y=0}^{y=b} dx$$

$$= \int_0^a \left[cxb + \frac{cb^2}{2} + \frac{c^2 b}{2} \right] dx$$

$$= \int_0^a \left[bcx + \frac{b^2 c}{2} + \frac{bc^2}{2} \right] dx$$

$$= \left[bc\frac{x^2}{2} + \frac{b^2 c}{2} x + \frac{bc^2}{2} x \right]_0^a$$

$$= \left[bc\frac{a^2}{2} + \frac{b^2 ca}{2} + \frac{bc^2 a}{2} \right] -$$

$$= \left[a^2 bc + ab^2 c + abc^2 \right] \frac{1}{2}$$

$$\therefore I = \frac{1}{2} \left[a^2 bc + ab^2 c + abc^2 \right]$$

$$I = \frac{abc}{2} [a+b+c]$$