Change of order of Integration

1) Change the order of Integration for $\int_0^1\int_0^x dx\,dy$

Soln:

Given Integral is not in the correct form.

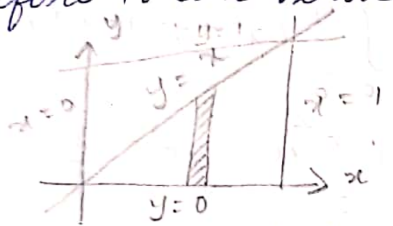
$$\int_0^1\int_0^x dx\,dy = \int_0^1\int_0^x dy\,dx$$

Given limits:

x limit : $x=0$ to $x=1$

y limit : $y=0$ to $y=x$

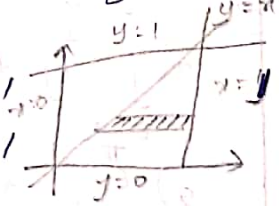
Inner limit is with respective y. Therefore it is a vertical strip.



By changing order of integration we have to draw a horizontal strip.

x limit : $x=y$ to $x=1$

y limit : $y=0$ to $y=1$



$$I = \int_0^1\int_y^1 dx\,dy$$

$\int_0^1\int_0^y dx\,dy = \int_0^1 [x]_{x=y}^{x=1} dy = \int_0^1 (1-y)\,dy = \left(y-\frac{y^2}{2}\right)_0^1 = \left(1-\frac{1}{2}\right)-(0-0) = \frac{1}{2}$

2) Change the order of Integration for $\int_0^1\int_0^y dx\,dy$

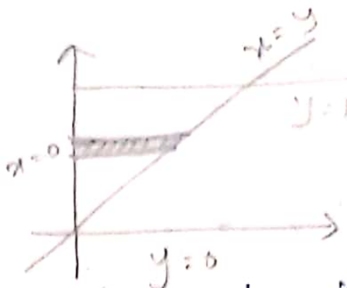
Soln:

Given Integral is in the correct form. $\int_0^1\int_0^y dx\,dy$.

Given limits

x limit : $x=0$ to $x=y$

y limit : $y=0$ to $y=1$



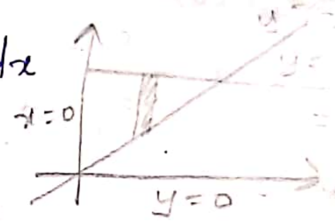
Inner limit is with respective to x, therefore it is a horizontal strip.

By changing order of integration. we have to draw a vertical strip

x limit : $x=0$ to $x=1$

y limit : $y=x$ to $y=1$

$$I = \int_0^1\int_x^1 dy\,dx$$



3) Evaluate by changing the order of integration $\int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$.

Soln:

Given Integral is in the correct form.

$$I = \int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$$

Given limits:

x limit : $x=0$ to $x=4$

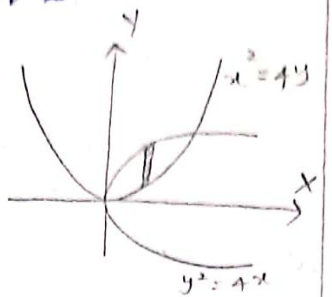
y limit : $y=\frac{x^2}{4}$ to $y=2\sqrt{x}$.

Inner limit is with respective to y. Therefore, it is a vertical strip.

$y = \frac{x^2}{4}$; $y = 2\sqrt{x}$

$4y = x^2$ — ①

$y^2 = 4x$ — ②



$x^2 = 4y$

squaring on both sides

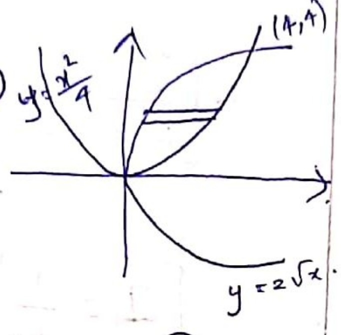
$x^4 = 16y^2$

$x^4 = 16(4x)$

$x^4 = 64x$

$x^3 = 64$

$\boxed{x = 4}$



sub $x = 4$ in eqn ②.

② $\Rightarrow y^2 = 4x$

$y^2 = 4(4)$

$y^2 = 16$

$\boxed{y = 4}$

By changing order of integration, we have to draw horizontal strip.

x limit : $x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$

y limit : $y = 0$ to $y = 4$

$$I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\, dy$$

$$= \int_0^4 [x]_{y^2/4}^{2\sqrt{y}} dy$$

$$= \int_0^4 \left[2\sqrt{y} - \frac{y^2}{4}\right] dy$$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right) dy$$

$$= \left[2\frac{y^{1/2+1}}{1/2+1} - \frac{y^3}{4\times 3}\right]_0^4$$

$$= \left[\frac{2y^{3/2}}{3/2} - \frac{y^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}(4^{3/2}) - \frac{1}{12}(4^3)\right] - [0]$$

$$= \left[\frac{4}{3}(4 \cdot 4^{1/2}) - \frac{1}{12}(4^3)\right]$$

$$= \frac{4}{3}(4\times 2) - \frac{1}{12}(64)$$

$$= \frac{32}{3} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3} \qquad \boxed{I = \frac{16}{3}}$$

4) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ using change of order of integration.

Solution:

Given integral is in correct form $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

Given limit

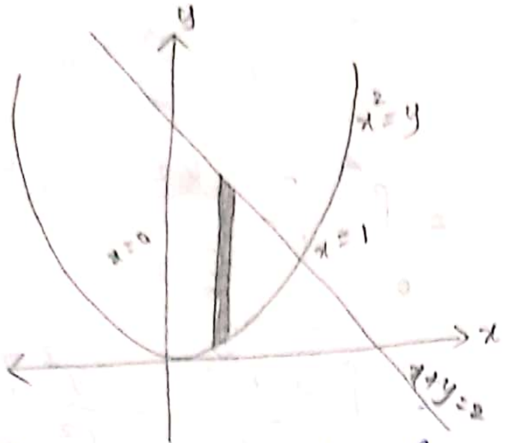
x limit: $x = 0$ to $x = 1$

y limit: $y = x^2$ to $y = 2 - x$

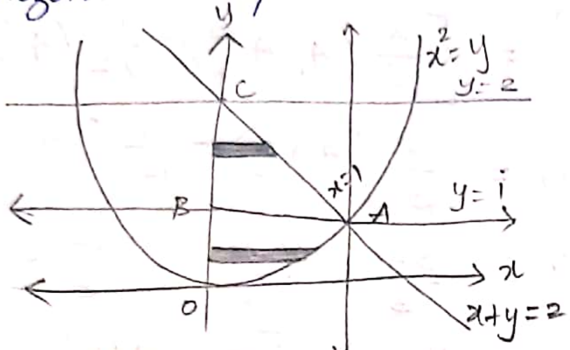
$\Rightarrow x + y = 2$

Inner limit is with respect to y.

Therefore it is a vertical strip.



By changing the order of integration we have to draw horizontal strip



As there is a parabola and straight line we have to draw two horizontal strips.

There are two regions.

(i) OAB (ii) ABC.

$\therefore$ I becomes $I = I_1 + I_2$.

In the region OAB,

x limit : $x=0$ to $x=\sqrt{y}$

y limit : $y=0$ to $y=1$

$$\therefore I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dy \, dx.$$

It is not in the correct form.

$$\therefore I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy.$$

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy$$

$$= \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 y \left[\frac{(\sqrt{y})^2}{2} - 0\right] dy$$

$$= \int_0^1 \frac{y \cdot y}{2} \, dy = \int_0^1 \frac{y^2}{2} \, dy$$

$$= \left[\frac{y^3}{2\times 3}\right]_0^1 = \frac{1}{6}(1^3 - 0)$$

$$= \frac{1}{6}.$$

$$\boxed{I_1 = \frac{1}{6}}$$

Now, in the region ABC

x limit; $x=0$ to $x=2-y$

y limit : $y=1$ to $y=2$

$$I_2 = \int_1^2 \int_0^{2-y} xy \, dy \, dx.$$

I_2 is not in the correct form.

$$I_2 = \int_1^2 \int_0^{2-y} xy \, dx \, dy.$$

$$= \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy.$$

$$= \int_1^2 y \left[\frac{(2-y)^2}{2} - 0\right] dy.$$

$$= \frac{1}{2}\int_1^2 y(4-4y+y^2)\,dy$$

$$= \frac{1}{2}\int_1^2 (4y-4y^2+y^3)\,dy.$$

$$= \frac{1}{2}\left[\frac{4y^2}{2}-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$= \left[y^2-\frac{2y^3}{3}+\frac{y^4}{8}\right]_1^2$$

$$= \left[2^2-\frac{2(2^3)}{3}+\frac{2^4}{8}\right]-\left[1-\frac{2}{3}+\frac{1}{8}\right]$$

$$= 4-\frac{16}{3}+\frac{16}{8}-1+\frac{2}{3}-\frac{1}{8}$$

$$= 3-\frac{14}{3}+\frac{15}{8} = \frac{72-112+45}{24}$$

$$\boxed{I_2 = \frac{5}{24}}$$

$$I = I_1 + I_2$$

$$= \frac{1}{6}+\frac{5}{24} = \frac{4+5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}.$$

$$\boxed{I = \frac{3}{8}}$$

5) Evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$ using change of order of integration.

Soln:

Given integration is in the correct form $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$

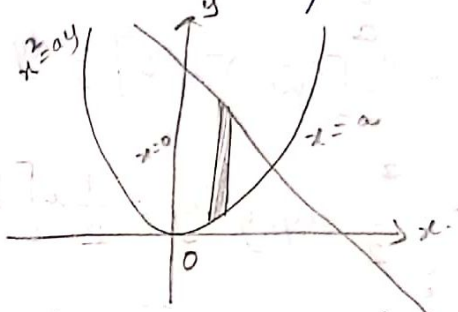
Given limit:

x limit : $x=0$ to $x=a$

y limit : $y=\frac{x^2}{a}$ to $y=2a-x$

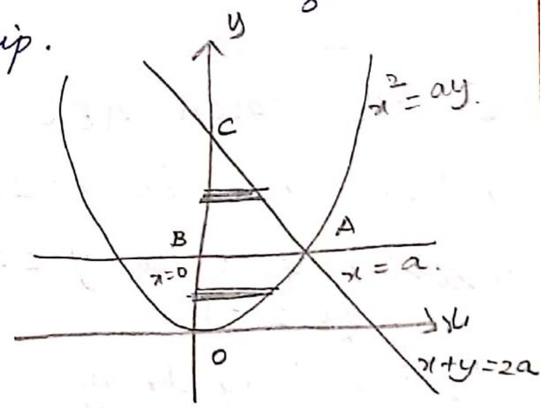
$x^2=ay \quad x+y=2a.$

Inner limit is with respective to y. Therefore, it is a vertical strip.



By changing the order of integration.

As there is a parabola and a straight line we have to draw two horizontal strip.



There are two regions (i) OAB (ii) ABC.

In the region OAB

x limit :

$x=0$ to $x=\sqrt{ay}$

y limit :

$y=0$ to $y=a$

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\,dy\,dx$$

I_1 is not in the correct form of integration.

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy.$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y \, dy.$$

$$= \int_0^a \frac{1}{2}\left[(\sqrt{ay})^2 - 0\right] y \, dy$$

$$= \int_0^a \frac{1}{2} ay.y \, dy = \frac{1}{2} a \left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{1}{2} a \left[\frac{a^3}{3} - 0\right]$$

$$= \frac{1}{2} a \left(\frac{a^3}{3}\right) = \frac{1}{6} a^4 = \frac{a^4}{6}.$$

$$\boxed{I_1 = \frac{a^4}{6}}$$

In the region ABC

x limit: $x=0$ to $x=2a-y$

y limit: $y=a$ to $y=2a$

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy$$

$$= \int_a^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y \, dy$$

$$= \int_a^{2a} \frac{(2a-y)^2}{2} y \, dy$$

$$= \frac{1}{2} \int_a^{2a} (4a^2 - 4ay + y^2) y \, dy$$

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$$= \frac{1}{2}\left\{\left[\frac{4a^2}{2}(2a)^2 - \frac{4a}{3}(2a)^3 + \frac{(2a)^4}{4}\right.\right.$$

$$\left.\left. - \left[\frac{4a^2}{2}(a^2) - \frac{4a}{3}(a^3) + \frac{a^4}{4}\right]\right\}\right.$$

$$= \frac{1}{2}\left\{\frac{4a^2}{2}4a^2 - \frac{4a}{3}(8a^3) + \frac{16a^4}{4}\right.$$

$$\left. - \frac{4a^4}{2} + \frac{4}{3}a^4 - \frac{a^4}{4}\right\}$$

$$= \frac{1}{2}\left\{\frac{16a^4}{2} - \frac{32a^4}{3} + \frac{16}{4}a^4\right.$$

$$\left. - \frac{4a^4}{2} + \frac{4}{3}a^4 - \frac{a^4}{4}\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{16}{2} - \frac{32}{3} + \frac{16}{4}\right)a^4\right.$$

$$\left. + \left(-\frac{4}{2} + \frac{4}{3} - \frac{1}{4}\right)a^4\right\}$$

$$= \frac{1}{2}\left\{\frac{12}{2} - \frac{28}{3} + \frac{15}{4}\right\} a^4$$

$$= \frac{1}{2}\left[\frac{144 - 224 + 90}{24}\right] a^4$$

$$= \frac{10a^4}{2 \times 24} = \frac{5a^4}{24}.$$

$$I_2 = \frac{5a^4}{24}$$

$$I = I_1 + I_2 = \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24}$$

$$\therefore \boxed{I = \frac{3a^4}{8}}$$

H.W

1) $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$ Ans: $\frac{a^4}{3}$.

2) $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ Ans: $\frac{1}{6} + \frac{5}{24} = \frac{3}{8}$

3) $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$ Ans: $\frac{64}{3} a^4$.

4) $\int_0^1 \int_y^{2-y} xy\, dx\, dy$ Ans: $\frac{1}{8} + \frac{5}{24} = \frac{1}{3}$.

5) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ Ans: 1.

change: $\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$.