Unit-I

Multiple Integrals

Introduction:

Applications of multiple Integrals are to find areas and volume of various bodies just by taking a little part of them into consideration.

Double Integration in Cartesian Co-ordinates

Problems:

1. Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$.

$$= \int_0^1 \int_1^2 (x^2 + xy)\,dy\,dx.$$

$$= \int_0^1 \left[x^2 \cdot y + x\frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^1 \left\{ \left[x^2(2) + x\frac{(2^2)}{2} \right] - \left[x^2 \cdot 1 + x\frac{(1^2)}{2} \right] \right\} dx.$$

$$= \int_0^1 \left\{ \left(2x^2 + x\frac{4}{2}\right) - \left(x^2 + \frac{x}{2}\right) \right\} dx.$$

$$= \int_0^1 \left\{ 2x^2 + 2x - x^2 - \frac{x}{2} \right\} dx$$

$$= \int_0^1 \left(x^2 + \frac{3x}{2}\right) dx = \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2} \right]_0^1$$

$$= \frac{1^3}{3} + \frac{3}{2}\left(\frac{1^2}{2}\right) - \left(\frac{0}{3} + \frac{3}{2}\left(\frac{0}{2}\right)\right) = \frac{1}{3} + \frac{3}{4} - 0 = \frac{4+9}{12}$$

$$= \frac{13}{12} \;//$$

2) Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dy\, dx$.

Solution:

$$\int_0^3 \int_0^2 e^{x+y}\, dy\, dx = \int_0^3 \int_0^2 e^x . e^y\, dy\, dx$$

$$= \int_0^3 e^x \left[e^y\right]_0^2 dx .$$

$$= \int_0^3 e^x \left[e^2 - e^0\right] dx$$

$$= \int_0^3 e^x \left[e^2 - 1\right] dx$$

$$= \int_0^3 (e^2 - 1) e^x\, dx$$

$$= (e^2 - 1) \int_0^3 e^x\, dx$$

$$= (e^2 - 1) \left[e^x\right]_0^3$$

$$= (e^2 - 1)(e^3 - e^0)$$

$$= (e^2 - 1)(e^3 - 1) .$$

3) Evaluate $\int_2^3 \int_1^2 \frac{dx\, dy}{xy}$

Note: $\log a - \log b = \log \frac{a}{b}$

Soln: $\int \frac{1}{x} dx = \log x + c$

$$\int_2^3 \int_1^2 \frac{dx\, dy}{xy} = \int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$$

$$= \int_2^3 \int_1^3 \frac{dx}{x} \frac{dy}{y}$$

$$= \int_2^3 \int_1^3 \left[\log x\right]_1^3 \frac{dy}{y} .$$

$$= \int_2^3 \left[\log 3 - \log 1\right] \frac{dy}{y}$$

$$= \int_2^3 \log\left(\frac{3}{1}\right) \frac{dy}{y}$$

$$= \int_2^3 (\log 3) \frac{dy}{y}$$

$$= \log 3 \int_2^3 \frac{dy}{y}$$

$$= \log 3 \left[\log y\right]_2^3$$

$$= \log 3 \left[\log 3 - \log 2\right]$$

$$= \log 3 \left[\log \frac{3}{2}\right] //$$

4) Evaluate

$$\int_0^5 \int_0^x x(x^2 + y^2)\, dx\, dy$$

Soln:

$$\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dx\, dy$$

$$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\, dx\, dy .$$

$$= \int_0^5 \int_0^{x^2} \left[x^3 + xy^2\right] dy\, dx .$$

$$= \int_0^5 \left[x^3 y + x \frac{y^3}{3}\right]_0^{x^2} dx .$$

$$= \int_0^5 \left\{\left[x^3 (x^2) + x \frac{(x^2)^3}{3}\right] - [0]\right\} dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^6}{6} + \frac{1}{3}\frac{x^8}{8}\right]_0^5$$

$$= \left[\frac{5^6}{6} + \frac{1}{3}\frac{5^8}{8}\right] - 0$$

$$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right]$$

$$= 5^6\left[\frac{4+25}{24}\right] = 5^6\left(\frac{27}{24}\right)$$

5) Evaluate $\int_0^a \int_0^b xy(x-y)\,dx\,dy$

Soln:

$$\int_0^a \int_0^b x^2 y^2 \quad \int_0^a \int_0^b xy(x-y)\,dx\,dy$$

$$= \int_0^a \int_0^b (x^2y - xy^2)\,dx\,dy$$

$$= \int_0^a \left[\frac{x^3}{3}y - \frac{x^2y^2}{2}\right]_0^b dy$$

$$= \int_0^a \left\{\left[\frac{b^3}{3}y - \frac{b^2y^2}{2}\right] - 0\right\} dy$$

$$= \int_0^a \left[\frac{b^3y}{3} - \frac{b^2y^2}{2}\right] dy$$

$$= \left[\frac{b^3y^2}{3\times 2} - \frac{b^2y^3}{2\times 3}\right]_0^a$$

$$= \left[\frac{b^3a^2}{6} - \frac{b^2a^3}{6}\right]$$

$$= \frac{1}{6}\left[b^3a^2 - b^2a^3\right]$$

$$= \frac{1}{6}\left(b^2a^2\right)(b-a)$$

$$= \frac{a^2b^2}{6}(b-a).$$

6) Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$

Soln:

$$\int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy = \int_0^a \left[\frac{x^2}{2}y\right]_0^{\sqrt{ay}} dy$$

$$= \int_0^a \left[\frac{(\sqrt{ay})^2}{2}y - 0\right] dy$$

$$= \int_0^a \frac{ay^2}{2}\,dy = \left[\frac{a}{2}\frac{y^3}{3}\right]_0^a$$

$$= \frac{a}{2}\left[\left(\frac{a^3}{3}\right) - (0)\right]$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) = \frac{a^4}{6}$$

1) $\int_0^1 \int_0^1 (x^2+y^2)\,dx\,dy$ Ans: $\frac{2}{3}$

2) $\int_0^1 \int_0^2 xy^2\,dy\,dx$ Ans: $\frac{4}{3}$.

3) $\int_0^a \int_0^b xy\,(x-y)\,dx\,dy$ Ans: $\frac{a^2b^2}{6}(b-a)$

4) $\int_0^1 \int_0^2 xy^2\,dy\,dx$ Ans: $\frac{4}{3}$.

5) $\int_0^{\pi} \int_0^x x \sin(x+y)\,dy\,dx$ Ans: -2

6) $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$ Ans: $3e^4 - 7$.

7) $\int_6^4 \int_6^{x^2} e^{y/x}\,dy\,dx$ Ans: $3e^4 - 7$

8) $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$ Ans: $\frac{a^4}{3}$

9) $\int_1^2 \int_2^3 xy^2\,dx\,dy$ Ans: $\frac{35}{6}$

10) $\int_0^3 \int_0^2 x\,dx\,dy$ Ans: 6

11) $\int_0^1 \int_0^{x^2} (x^2+y)\,dy\,dx$ Ans: $\frac{26}{105}$