Euler's method.

$$y_{n+1} = y_n + h f(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

Modified Euler's method

$$y_{n+1} = y_n + h\left[ f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

4. Compute $y$ at $x = 0.25$ by modified Euler's method given $y' = 2xy$, $y(0)=1$

$f(x,y) = 2xy$

$x_0 = 0$, $y_0 = 1$, $x_1 = 0.25$, $h = 0.25 - 0 = 0.25$

By modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 1 + (0.25) f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2} 2x_0y_0\right)$$

$$= 1 + (0.25) f(0.125, 1 + (0.125)(2(0)(1)))$$

$$= 1 + (0.25) f(0.125, 1)$$

$$= 1 + (0.25)\, 2(0.125)(1)$$

$$= 1 + 0.0625$$

$$= 1.0625$$

5. Using modified Euler's method, compute $y(0.1)$ with $h=0.1$, from $y' = y - \frac{2x}{y}$, $y(0)=1$.

Given: $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1$

$f(x,y) = y - \frac{2x}{y}$.

By modified Euler's method,

$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))$

$y_1 = y(0.1) = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$

$= 1 + 0.1 f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(y_0 - \frac{2x_0}{y_0}))$

$= 1 + 0.1 f(0.05, 1 + (0.05)(1-0))$

$= 1 + 0.1 f(0.05, 1.05)$

$= 1 + (0.1)(1.05 - \frac{2(0.05)}{1.05})$

$= 1 + (0.1)(0.9548)$

$= 1 + 0.09548$

$= 1.09548.$

6. Using modified Euler's method find $y(0.1)$, if $\frac{dy}{dx} = x^2 + y^2$

$y(0)=1$. Given: $f(x,y) = x^2 + y^2$

$x_0 = 0$, $y_0 = 1$, $h = 0.1 - 0 = 0.1$, $x_1 = 0.1$

$y_1 = y(0.1) = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$

$= 1 + (0.1) f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(0^2 + (0.1)^2))$

$= 1 + (0.1) f(0.05, 1 + (0.05)(0.01))$

$= 1 + (0.1) f(0.05, 1.0005)$

$= 1 + (0.1)[(0.05)^2 + (1.0005)^2]$

$= 1 + (0.1)(1.0035)$

$= 1.10035$