Numerical differentiation and Numerical integration.

Numerical differentiation:

It is the process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x, from the given data (x_i, y_i). If the values of x are equally spaced we can use Newton's interpolation formula for equal intervals. If the value of x are unequally spaced we can use Lagrange's interpolation formula (or) Newton's divided difference interpolation formula.

Differentiation using interpolation formula:

Newton's forward difference formula to compute the derivatives:

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$

$$\text{(ie)}\ y = y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + \cdots \quad -(1)$$

Now, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}.$$

$$\text{(ie)}\ \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right\} \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right] \quad -(3)$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\right] \quad -(4).$$

In particular at $x = x_0$, $u = 0$. Hence put $u=0$ in (2), (3), (4) we get the values of I, II, III derivatives at $x = x_0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's backward formula to compute the derivatives.

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots\right]$$

In particular at $x = x_n$, $v = 0$, then

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$