Newton's forward and backward difference formula.

Let the function $y=f(x)$ takes the values $y_0, y_1, \ldots, y_n$ at the points $x_0, x_1, \ldots, x_n$ where $x_i = x_0 + ih$. Then Newton's Forward interpolation polynomial is given by,

$$y(x) = P_n(x) = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)\cdots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x-x_0}{h}$, h = difference between two intervals.

Then Newton's backward interpolation polynomial is given by,

$$y(x) = P_n(x) = f(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots + \frac{u(u+1)\cdots(u+(n-1))}{n!}\nabla^n y_n$$

where $u = \frac{x-x_n}{h}$,

Note: Forward	Backward.
First order:	First order:
$\Delta y_0 = y_1 - y_0$	$\nabla y_n = y_n - y_{n-1}$
$\Delta y_1 = y_2 - y_1$	$\nabla y_{n-1} = y_{n-1} - y_{n-2}$.
Second order	Second order
$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$.	$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$
Third order.	Third order
$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	$\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$

1. Using Newton's forward interpolation formula find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$

x	4	6	8	10
y	1	3	8	10

Also find backward interpolation.

We form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
			3	
6	3			-6
		5		
8	8		-3	
		2		
10	10			

Forward interpolation:

Here, $x_0 = 4$, $y_0 = 1$, $h = 6-4 = 2$.

$$u = \frac{x - x_0}{h} = \frac{x-4}{2}.$$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 1 + \frac{x-4}{2}\cdot\frac{2}{1!} + \frac{(x-4)}{2}\left[\frac{x-4}{2} - 1\right]\frac{3}{2!} + \frac{x-4}{2}\left[\frac{x-4}{2} - 1\right]\left[\frac{x-4}{2} - 2\right]\frac{(-6)}{3!}$$

$$= 1 + x - 4 + \frac{(x-4)(x-6)\,3}{8} + \frac{(-)(x-4)(x-6)(x-8)}{8}$$

$$= \frac{8 + (x-4)8 + 3(x^2 - 10x + 24) - (x^3 - 18x^2 + 104x - 192)}{8}$$

$$y(x) = \frac{1}{8}\left[-x^3 + 21x^2 - 126x + 240\right]$$

$$\therefore y(5) = \frac{1}{8}\left[-5^3 + 21(5)^2 - 126(5) + 240\right]$$

$$= \frac{1}{8}\left[-125 + 525 - 630 + 240\right]$$

$$= \frac{1}{8}[10] = \frac{5}{4} = 1.25$$

Backward interpolation:

Here, $x_n = 10$, $y_n = 10$, $h = 10 - 8 = 2$. $\Rightarrow u = \frac{x-10}{2}$.

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n.$$

$$= 10 + \frac{(x-10)}{2}\cdot\frac{2}{1!} + \left(\frac{x-10}{2}\right)\left[\frac{x-10}{2} + 1\right]\frac{(-3)}{2!} + \left(\frac{x-10}{2}\right)\left(\frac{x-10}{2}+1\right)\left(\frac{x-10}{2}+2\right)\left(\frac{-6}{3!}\right)$$

$$= 10 + x - 10 \cancel{+} - \frac{(x-10)(x-8)(3)}{8} - \frac{(x-10)(x-8)(x-6)}{8}$$

$$= \frac{1}{8}\left[80 + 8x - 80 - 3x^2 + 54x - 240 - x^3 + 24x^2 - 188x + 480\right]$$

$$y(x) = \frac{1}{8}\left[-x^3+21x^2-126x+240\right]$$

2. Using Newton's forward interpolation formula find the polynomial $f(x)$ satisfying the following data. Hence find $f(2)$.

x	0	5	10	15
y	14	379	1444	3584

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	14			
		365		
5	379		700	
		1065		375
10	1444		1075	
		2140		
15	3584			

Forward interpolation:

$x_0 = 0,\ y_0 = 14,\ h = 5-0 = 5$

$$u = \frac{x-0}{5} = \frac{x}{5}$$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 14 + \frac{x}{5}\,\frac{365}{1!} + \frac{x}{5}\left(\frac{x}{5}-1\right)\frac{700}{2!} + \frac{x}{5}\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\frac{375}{3!}$$

$$= 14 + x(73) + \frac{x}{5}\left(\frac{x-5}{5}\right)350 + \frac{x}{5}\left(\frac{x-5}{5}\right)\left(\frac{x-10}{5}\right)\left(\frac{375}{6}\right)$$

$$= 14 + 73x + 14x^2 - 70x + \frac{x^3-15x^2+50x}{2}$$

$$= \frac{1}{2}\left(28 + 146x + 28x^2 - 140x + x^3 - 15x^2 + 50x\right)$$

$$= \frac{1}{2}\left(x^3 + 13x^2 + 56x + 28\right)$$

$$f(2) = \frac{1}{2}\left(2^3 + 13(2)^2 + 56(2) + 28\right)$$

$$= \frac{1}{2}\left(8 + 52 + 112 + 28\right)$$

$$= \frac{1}{2}(200) = 100.$$