MODULE –III

DESIGN OF SPRINGS

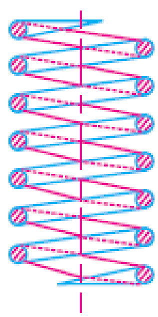
Spring

A spring is defined as an elastic body, whose function is to distort when loaded and to recover its original shape when the load is removed.

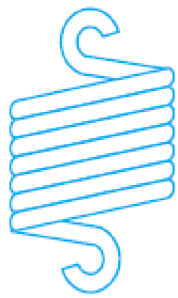
Types of springs

Helical springs

The helical springs are made up of a wire coiled in the form of a helix and are primarily intended for compressive or tensile loads. The cross-section of the wire from which the spring is made may be circular, square or rectangular. The two forms of helical springs are compression helical spring and tension helical spring. The major stresses produced in helical springs are shear stresses due to twisting.

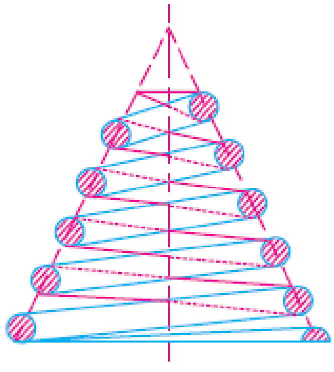


(*a*) Compression helical spring.

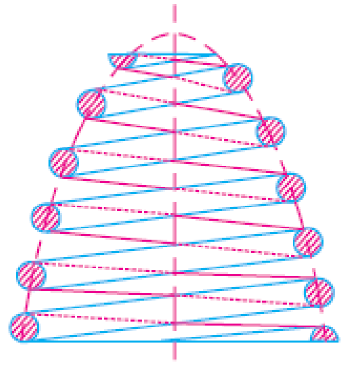


(*b*) Tension helical spring.

Conical and volute springs



(*a*) Conical spring.

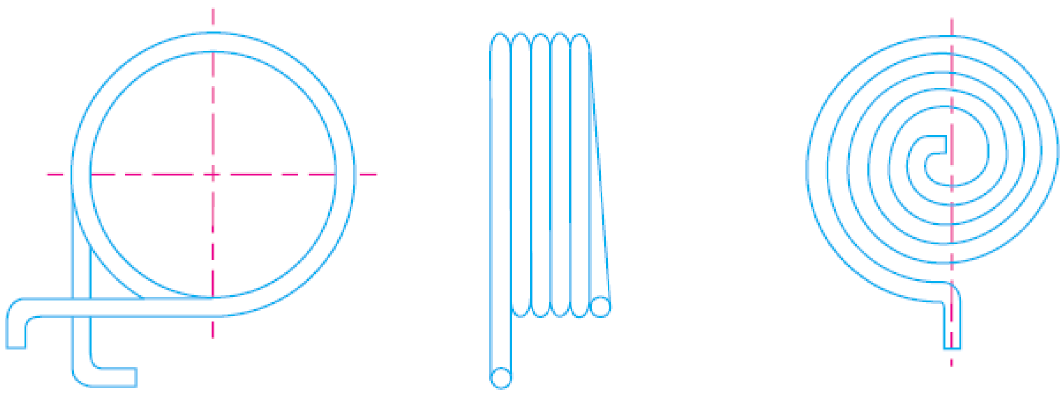


(*b*) Volute spring.

The conical and volute springs, are used in special applications where a telescoping spring or a spring with a spring rate that increases with the load is desired. The major stresses produced in conical and volute springs are also shear stresses due to twisting.

Torsion springs

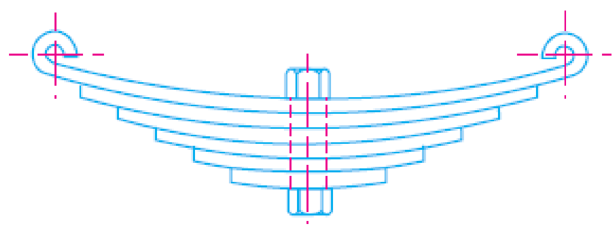
These springs may be of helical or spiral type. The major stresses produced in torsion springs are tensile and compressive due to bending.



(*a*) Helical torsion spring. (*b*) Spiral torsion spring.

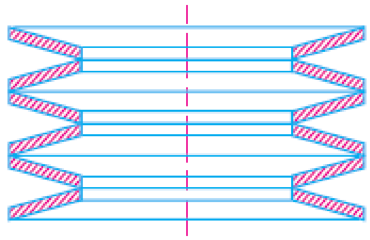
Laminated or leaf springs

The laminated or leaf spring or flat spring or carriage spring consists of a number of flat plates known as leaves of varying lengths held together by means of clamps and bolts. These are mostly used in automobiles. The major stresses produced in leaf springs are tensile and compressive stresses.



Disc or bellevile springs

These springs consist of a number of conical discs held together against slipping by a central bolt or tube. These springs are used in applications where high spring rates and compact spring units are required. The major stresses produced in disc or bellevile springs are tensile and compressive stresses.

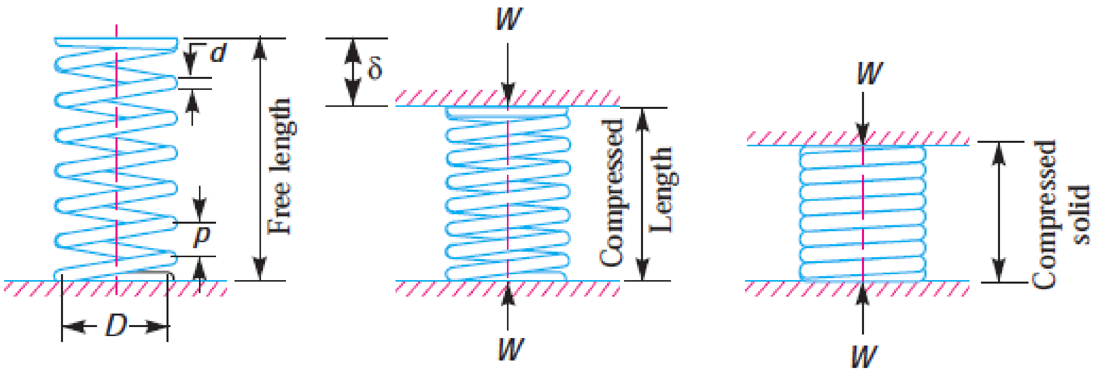


Material for Springs

The springs are mostly made from oil-tempered carbon steel wires containing 0.60 to 0.70 percent carbon and 0.60 to 1.0 percent manganese.

Terminology

The following terms used in connection with compression springs are important from the subject point of view.



Solid length

When the compression spring is compressed until the coils come in contact with each other, then the spring is said to be solid. The solid length of a spring is the product of total number of coils and the diameter of the wire. Mathematically,

Solid length of the spring, $L_S = n'.d$

where,

n' = Total number of coils
d = Diameter of the wire

Free length

The free length of a compression spring is the length of the spring in the free or unloaded condition. It is equal to the solid length plus the maximum deflection or compression of the spring and the clearance between the adjacent coils when fully compressed. Clearance may be taken as 15 percent of the maximum deflection. Mathematically,

Free length of the spring,

L_F = Solid length + Maximum compression + Clearance between adjacent coils

$= n'.d + \delta_{max} + 0.15\ \delta_{max}$

Spring index

The spring index is defined as the ratio of the mean diameter of the coil to the diameter of the wire. Mathematically,

Spring index, $C = D / d$

where,

D = Mean diameter of the coil

d = Diameter of the wire

Spring rate

The spring rate or stiffness or spring constant is defined as the load required per unit deflection of the spring. Mathematically,

Spring rate, $k = W / \delta$

where,

W = Load

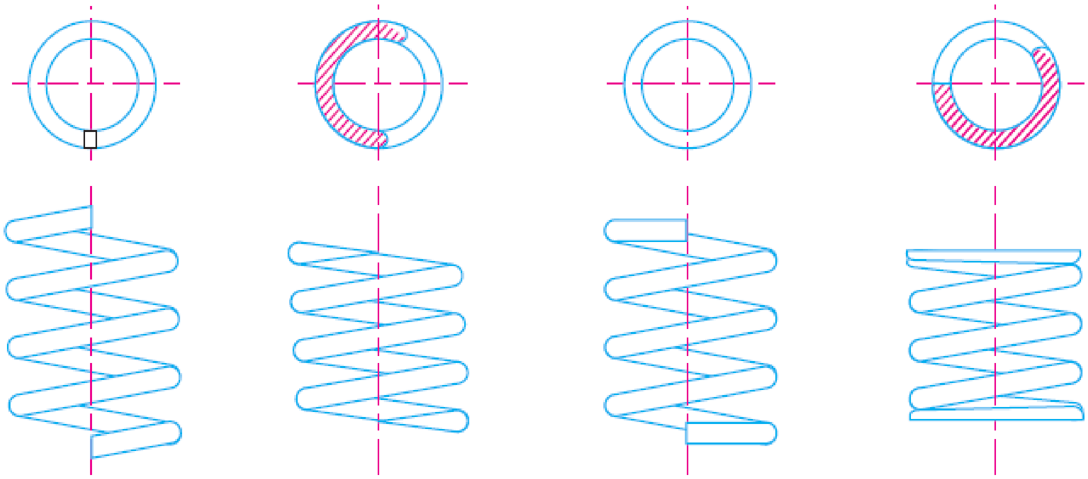
δ = Deflection of the spring

Pitch

The pitch of the coil is defined as the axial distance between adjacent coils in uncompressed state. Mathematically, Pitch of the coil,

$$p = \frac{\text{Free length}}{n' - 1}$$

End Connections for Compression Helical Springs



(*a*) Plain ends. (*b*) Ground ends. (*c*) Squared ends. (*d*) Squared and ground ends.

It may be noted that part of the coil which is in contact with the seat does not contribute to spring action and hence are termed as inactive coils. The turns which impart spring action are known as active turns. As the load increases, the number of inactive coils also increases due to seating of the end coils and the amount of increase varies from 0.5 to 1 turn at the usual working loads. The following table shows the total number of turns, solid length and free length for different types of end connections.

Type of end	*Total number of turns (n')*	*Solid length*	*Free length*
1. Plain ends	n	$(n + 1)\,d$	$p \times n + d$
2. Ground ends	n	$n \times d$	$p \times n$
3. Squared ends	$n + 2$	$(n + 3)\,d$	$p \times n + 3d$
4. Squared and ground ends	$n + 2$	$(n + 2)\,d$	$p \times n + 2d$

where,

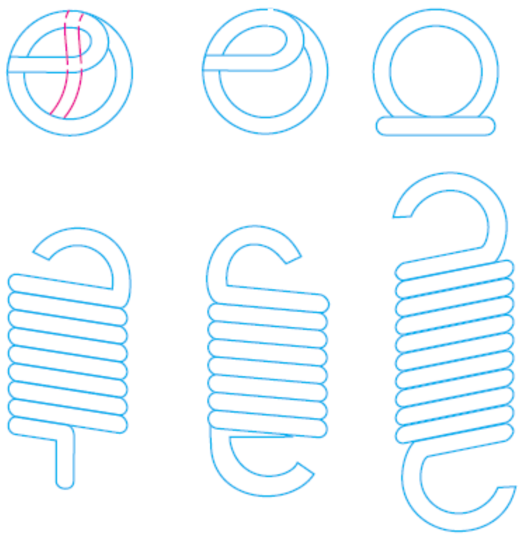
n = Number of active turns

p = Pitch of the coils

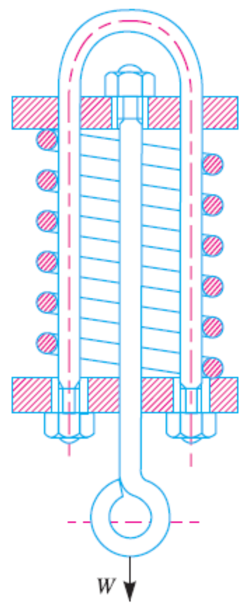
d = Diameter of the spring wire

End Connections for Tension Helical Springs

The tensile springs are provided with hooks or loops as shown in figure. These loops may be made by turning whole coil or half of the coil. In a tension spring, large stress concentration is produced at the loop.



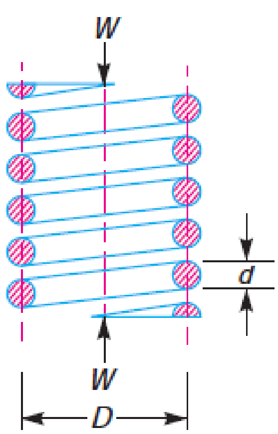
End connection for tension helical springs.



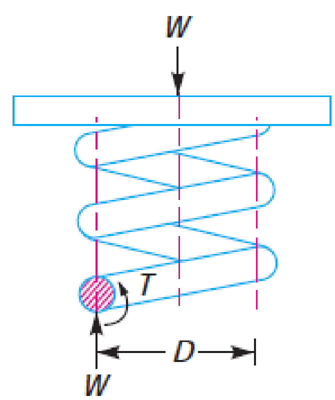
Compression spring for carrying tensile load.

Stresses in Helical Springs of Circular Wire

Consider a helical compression spring made of circular wire and subjected to an axial load *W*, as shown in figure.



(*a*) Axially loaded helical spring.



(*b*) Free body diagram showing that wire is subjected to torsional shear and a direct shear.

Let,

D = Mean diameter of the spring coil
d = Diameter of the spring wire
n = Number of active coils
G = Modulus of rigidity for the spring material
W = Axial load on the spring
τ = Maximum shear stress induced in the wire
C = Spring index = D/d
p = Pitch of the coils
δ = Deflection of the spring, as a result of an axial load W

Maximum shear stress induced in the wire,

$$\tau = K \times \frac{8\,W.D}{\pi\, d^3} = K \times \frac{8\,W.C}{\pi\, d^2}$$

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

Deflection of Helical Springs of Circular Wire

Stiffness of the spring or spring rate,

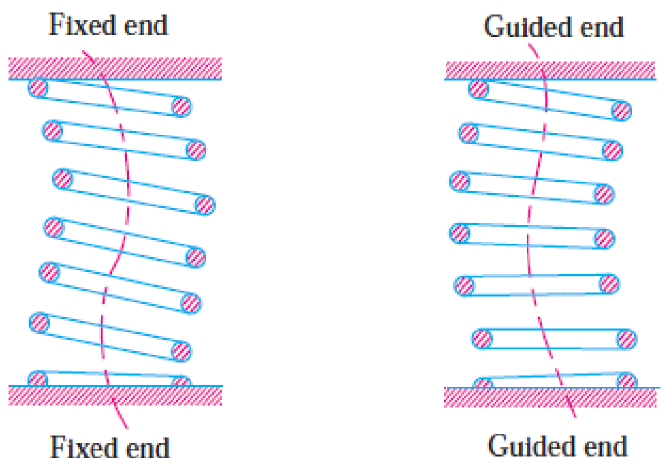
$$\frac{W}{\delta} = \frac{G.d^4}{8\,D^3.n} = \frac{G.d}{8\,C^3.n} = \text{constant}$$

Eccentric Loading of Springs

Sometimes, the load on the springs does not coincide with the axis of the spring, *i.e.* the spring is subjected to an eccentric load. In such cases, not only the safe load for the spring reduces, the stiffness of the spring is also affected. The eccentric load on the spring increases the stress on one side of the spring and decreases on the other side. When the load is offset by a distance e from the spring axis, then the safe load on the spring may be obtained by multiplying the axial load by the factor $\frac{D}{2\,e + D}$, where D is the mean diameter of the spring.

Buckling of Compression Springs

It has been found experimentally that when the free length of the spring (L_F) is more than four times the mean or pitch diameter (D), then the spring behaves like a column and may fail by buckling at a comparatively low load as shown in figure.



The critical axial load (W_{cr}) that causes buckling may be calculated by using the following relation,

$W_{cr} = k \times K_B \times L_F$

where,

k = Spring rate or stiffness of the spring = W/δ

L_F = Free length of the spring

K_B = Buckling factor depending upon the ratio L_F / D

Surge in Springs

When one end of a helical spring is resting on a rigid support and the other end is loaded suddenly, then all the coils of the spring will not suddenly deflect equally, because some time is required for the propagation of stress along the spring wire. A little consideration will show that in the beginning, the end coils of the spring in contact with the applied load takes up whole of the deflection and then it transmits a large part of its deflection to the adjacent coils. In this way, a wave of compression propagates through the coils to the supported end from where it is reflected back to the deflected end. This wave of compression travels along the spring indefinitely. If the applied load is of fluctuating type as in the case of valve spring in internal combustion engines and if the time interval between the load applications is equal to the time required for the wave to travel from one end to the other end, then resonance will occur. This results in very large deflections of the coils and correspondingly very high stresses. Under these conditions, it is just possible that the spring may fail. This phenomenon is called surge. It has been found that the natural frequency of spring should be atleast twenty times the frequency of application of a periodic load in order to avoid resonance. The natural frequency for springs clamped between two plates is given by

$$f_n = \frac{d}{2\pi D^2 . n} \sqrt{\frac{6\,G.g}{\rho}} \text{ cycles/s}$$

where,

d = Diameter of the wire
D = Mean diameter of the spring
n = Number of active turns
G = Modulus of rigidity
g = Acceleration due to gravity
ρ = Density of the material of the spring

Problem 1

A mechanism used in printing machinery consists of a tension spring assembled with a preload of 30 N. The wire diameter of spring is 2 mm with a spring index of 6. The spring has 18 active coils. The spring wire is hard drawn and oil tempered having following material properties:

- Design shear stress = 680 MPa
- Modulus of rigidity = 80 kN/mm^2

Determine:

- Initial torsional shear stress in the wire
- Spring rate
- The force to cause the body of the spring to its yield strength.

Given

W_i = 30 N, d = 2 mm, $C = D/d$ = 6, n = 18, τ = 680 MPa, G = 80 kN/mm^2

To Find

τ_i, Spring rate, W

Soln

Wahl's stress factor,

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C} = 1.2525$$

Initial torsional shear stress in the wire,

$$\tau_i = K \times \frac{8\,W_i \times C}{\pi\,d^2} = 1.2525 \times \frac{8 \times 30 \times 6}{\pi \times 2^2} = 143.5 \text{ N/mm}^2$$

Spring rate or stiffness of the spring,

$$= \frac{G.d}{8\,C^3.n} = \frac{80 \times 10^3 \times 2}{8 \times 6^3 \times 18} = 5.144 \text{ N/mm}$$

Let,

W = Force to cause the body of the spring to its yield strength.

Design or maximum shear stress (τ),

$$680 = K \times \frac{8\,W.C}{\pi\,d^2} \rightarrow W = 142.25 \text{ N}$$

Problem 2

Design a helical compression spring for a maximum load of 1000 N for a deflection of 25 mm using the value of spring index as 5. The maximum permissible shear stress for spring wire is 420 MPa and modulus of rigidity is 84 kN/mm^2. Take Wahl's factor,

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

where,

C = Spring index.

Given

W = 1000 N, δ = 25 mm, C = D/d = 5, τ = 420 MPa, G = 84 kN/mm^2

To Find

Design a helical compression spring

Soln

Let,

D = Mean diameter of the spring coil
d = Diameter of the spring wire

Wahl's stress factor,

$$K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = 1.31$$

Maximum shear stress (τ),

$$420 = K \times \frac{8\,W.C}{\pi\, d^2} \rightarrow d = 6.3 \text{ mm}$$

A standard wire of size *SWG* 3 having diameter (*d)* = 6.401 mm is selected from PSG DDB PG.NO: 13.1.

Mean diameter of the spring coil,

$$D = C.d = 5\,d = 5 \times 6.401 = 32.005 \text{ mm}$$

Outer diameter of the spring coil,

$$D_o = D + d = 32.005 + 6.401 = 38.406 \text{ mm}$$

Let

n = Number of active turns of the coils

Deflection of the spring (δ),

$$25 = \frac{8W.C^3.n}{G.d} \rightarrow n = 13.44 \text{ say } 14$$

Assuming squared and ground ends, the total number of turns,

$$n' = n + 2 = 16$$

Free length of the spring,

$$L_F = n'.d + \delta + 0.15\,\delta = 131.2 \text{ mm}$$

Pitch of the coil,

$$= \frac{\text{Free length}}{n' - 1} = 8.75 \text{ mm}$$

Energy Stored in Helical Springs of Circular Wire

Let,

W = Load applied on the spring

δ = Deflection produced in the spring due to the load W

Assuming that the load is applied gradually, the energy stored in a spring is,

$$U = \frac{1}{2} W.\delta \qquad \text{... } \textbf{(}\boldsymbol{i}\textbf{)}$$

Substituting,

$$\tau = K \times \frac{8\,W.D}{\pi\,d^3} \text{ or } W = \frac{\pi\,d^3.\tau}{8\,K.D}$$

And

$$\delta = \frac{8\,W.D^3.n}{G.d^4}$$ in equation ***(i)*** and reducing

$$U = \frac{\tau^2}{4\,K^2.G} \times V$$

where,

V = Volume of the spring wire

= Length of spring wire × Cross-sectional area of spring wire

$$= (\pi\,D.n)\left(\frac{\pi}{4} \times d^2\right)$$

Problem 3

A closely coiled helical spring is made of 10 mm diameter steel wire, the coil consisting of 10 complete turns with a mean diameter of 120 mm. The spring carries an axial pull of 200 N. Determine the shear stress induced in the spring. Determine also the deflection in the spring, its stiffness and strain energy stored by it if the modulus of rigidity of the material is 80 kN/mm^2.

Given

d = 10 mm, n = 10, D = 120 mm, W = 200 N, G = 80 kN/mm^2

To Find

τ, δ, Stiffness, U

Soln

Shear stress induced in the spring,

$$\tau = K \times \frac{8\,W \times C}{\pi\,d^2} = 68.4 \text{ MPa}$$

Where,

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C} \quad \text{and } C = D/d$$

Deflection of the spring,

$$\delta = \frac{8\,W.D^3 n}{G.d^4} = 34.56 \text{ mm}$$

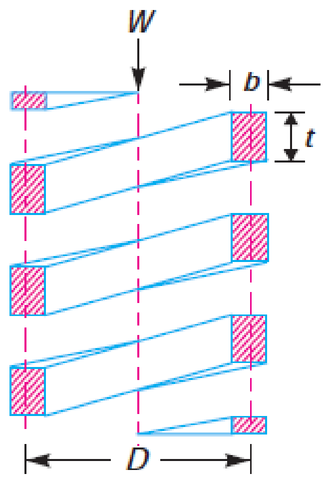
Stiffness of the spring,

$$= \frac{W}{\delta} = \frac{200}{34.56} = 5.8 \text{ N/mm}$$

Strain energy stored in the spring,

$$U = \frac{1}{2}\,W.\delta = 3.456 \text{ N-m}$$

Stress and Deflection in Helical Springs of Non-circular Wire



Shear stress, deflection and stiffness formulas are available in PSG DDB PG.NO: 7.100.

Problem 4

A loaded narrow-gauge car of mass 1800 kg and moving at a velocity 72 m/min., is brought to rest by a bumper consisting of two helical steel springs of square section. The mean diameter of the coil is six times the side of the square section. In bringing the car to rest, the springs are to be compressed by 200 mm. Assuming the allowable shear stress as 365 MPa and spring index of 6, find:

- Maximum load on each spring
- Side of the square section of the wire
- Mean diameter of coils
- Number of active coils

Take modulus of rigidity as 80 kN/mm^2.

Given

m = 1800 kg, ν = 72 m/min = 1.2 m/s, δ = 200 mm, τ = 365 MPa, C = 6, G = 80 kN/mm^2

To Find

W, b, D, n

Soln

Let,

W = Maximum load on each spring

Kinetic energy of the car,

$$= \frac{1}{2} m.v^2 = \frac{1}{2} \times 1800\ (1.2)^2 = 1296 \text{ N-m}$$

This energy is absorbed in the two springs when compressed to 200 mm. If the springs are loaded gradually from 0 to W, then

Energy absorbed by the spring = Load × Deflection = Kinetic energy of the car

$$\left(\frac{0 + W}{2}\right) 2 \times 200 = 1296 \times 10^3$$

$\rightarrow W = 6480$ N(multiplied by 2 since 2 springs)

Let,

b = Side of the square section of the wire

D = Mean diameter of the coil = 6 b ... ($C = D/b = 6$)

Wahl's stress factor,

$$K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = 1.2525$$

From maximum shear stress (τ) formula, b = 16 mm.

Mean diameter of the coil,

$D = 6\ b = 6 \times 18 = 96$ mm

Let,

n = Number of active coils

From deflection formula, $n = 33$

Helical Springs Subjected to Fatigue Loading

Problem 5

A helical compression spring made of oil tempered carbon steel is subjected to a load which varies from 400 N to 1000 N. The spring index is 6 and the design factor of safety is 1.25. If the yield stress in shear is 770 MPa and endurance stress in shear is 350 MPa, find:

- Size of the spring wire
- Diameters of the spring
- Number of turns of the spring
- Free length of the spring

The compression of the spring at the maximum load is 30 mm. The modulus of rigidity for the spring material may be taken as 80 kN/mm^2.

Given

W_{min} = 400 N, W_{max} = 1000 N, C = 6, F.S. = 1.25, τ_y = 770 MPa, τ_e = 350 MPa, δ = 30 mm, G = 80 kN/mm^2

To Find

d, D, D_o, D_i, n, n', L_F

Soln

Let,

d = Diameter of the spring wire

D = Mean diameter of the spring = *C.d* = 6 *d*

Mean load,

$$W_m = \frac{W_{max} + W_{min}}{2} = \frac{1000 + 400}{2} = 700 \text{ N}$$

Variable load,

$$W_v = \frac{W_{max} - W_{min}}{2} = \frac{1000 - 400}{2} = 300 \text{ N}$$

Wahl's stress factor,

$$K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = 1.2525$$

Direct shear factor, $K_s = K/K_c = 1.083$

Mean shear stress,

$$\tau_m = K_S \times \frac{8\, W_m \times D}{\pi\, d^3} = \frac{11\,582}{d^2} \text{ N/mm}^2$$

Variable shear stress,

$$\tau_v = K \times \frac{8\, W_v \times D}{\pi\, d^3} = \frac{5740}{d^2} \text{ N/mm}^2$$

WKT,

$$\frac{1}{F.S.} = \frac{\tau_m - \tau_v}{\tau_y} + \frac{2\,\tau_v}{\tau_e} \rightarrow d = 7.1 \text{ mm}$$

Mean diameter of the spring,

$D = C.d = 6 \times 7.1 = 42.6$ mm

Outer diameter of the spring,

$D_o = D + d = 42.6 + 7.1 = 49.7$ mm

Inner diameter of the spring,

$D_i = D - d = 42.6 - 7.1 = 35.5$ mm

Let,

n = Number of active turns of the spring

Deflection of the spring (δ),

$$30 = \frac{8\, W.D^3.n}{G.d^4} \rightarrow n = 9.87 \text{ say } 10$$

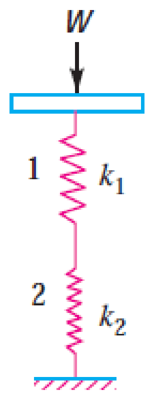
Assuming the ends of the spring to be squared and ground, the total number of turns of the spring,

$n' = n + 2 = 10 + 2 = 12$

Free length of the spring,

$L_F = n'.d + \delta + 0.15\,\delta = 119.7$ say 120 mm

Springs in Series



Consider two springs connected in series as shown in figure.

Let,

W = Load carried by the springs
δ_1 = Deflection of spring 1
δ_2 = Deflection of spring 2
k_1 = Stiffness of spring 1 = W / δ_1
k_2 = Stiffness of spring 2 = W / δ_2

When the springs are connected in series, then the total deflection produced by the springs is equal to the sum of the deflections of the individual springs.

Total deflection of the springs,

$$\delta = \delta_1 + \delta_2$$

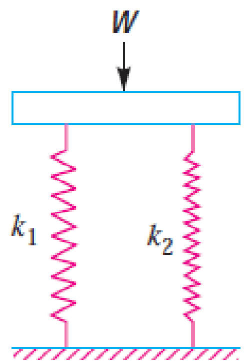
$$\frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2}$$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

where,

k = Combined stiffness of the springs

Springs in Parallel



Consider two springs connected in parallel as shown in figure.

Let,

W = Load carried by the springs
W_1 = Load shared by spring 1
W_2 = Load shared by spring 2
k_1 = Stiffness of spring 1
k_2 = Stiffness of spring 2

When the springs are connected in parallel, then the total deflection produced by the springs is same as the deflection of the individual springs.

WKT,

$$W = W_1 + W_2$$

$$\delta.k = \delta.k_1 + \delta.k_2$$

$$k = k_1 + k_2$$

where,

k = Combined stiffness of the springs
δ = Deflection produced

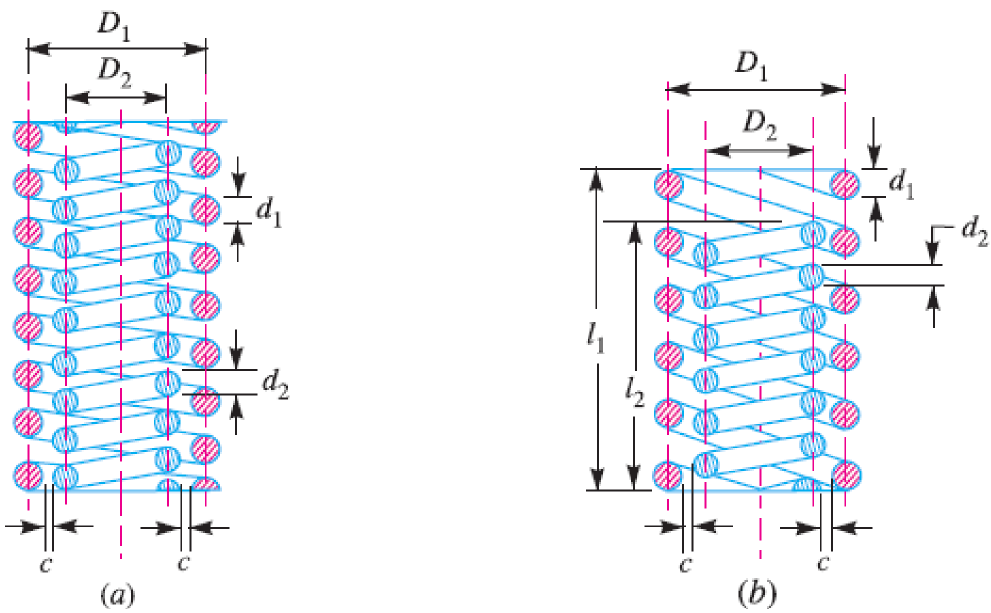
Concentric or Composite Springs

Consider a concentric spring as shown in figure.

Let,

W = Axial load
W_1 = Load shared by outer spring
W_2 = Load shared by inner spring

d_1 = Diameter of spring wire of outer spring
d_2 = Diameter of spring wire of inner spring
D_1 = Mean diameter of outer spring
D_2 = Mean diameter of inner spring
δ_1 = Deflection of outer spring
δ_2 = Deflection of inner spring
n_1 = Number of active turns of outer spring
n_2 = Number of active turns of inner spring



Assuming that both the springs are made of same material, then the maximum shear stress induced in both the springs is approximately same,

$$\tau_1 = \tau_2$$

If both the springs are effective throughout their working range, then their free length and deflection are equal,

$$\delta_1 = \delta_2$$

When both the springs are compressed until the adjacent coils meet, then the solid length of both the springs is equal,

$$n_1.d_1 = n_2.d_2$$

Based on the above mentioned relations, the following expressions are obtained,

$$\frac{D_1}{d_1} = \frac{D_2}{d_2} = C, \text{ the spring index}$$

$$\frac{W_1}{W_2} = \frac{(d_1)^2}{(d_2)^2}$$

$$\frac{d_1}{d_2} = \frac{C}{C-2}$$

$$\frac{D_1 - D_2}{2} = d_1$$

Radial clearance between the two springs,

$$c = \left(\frac{D_1}{2} - \frac{D_2}{2}\right) - \left(\frac{d_1}{2} + \frac{d_2}{2}\right) = \frac{d_1 - d_2}{2}$$

Problem 6

A concentric spring for an aircraft engine valve is to exert a maximum force of 5000 N under an axial deflection of 40 mm. Both the springs have same free length, same solid length and are subjected to equal maximum shear stress of 850 MPa. If the spring index for both the springs is 6, find

- The load shared by each spring
- The main dimensions of both the springs
- The number of active coils in each spring

Assume G = 80 kN/mm^2 and diametral clearance to be equal to the difference between the wire diameters.

Given

W = 5000 N, δ = 40 mm, $L_{F1} = L_{F2}$, $L_{S1} = L_{S2}$, $\tau_1 = \tau_2$ = 850 MPa, C = 6, G = 80 kN/mm^2

To Find

W_1, W_2, Main dimensions of the springs, n_1, n_2

Soln

$$\frac{d_1}{d_2} = \frac{C}{C-2} = \frac{6}{6-2} = 1.5$$

$$\frac{W_1}{W_2} = \left(\frac{d_1}{d_2}\right)^2 = (1.5)^2 = 2.25 \qquad \text{... }(\textbf{\textit{i}})$$

$$W_1 + W_2 = W = 5000 \text{ N} \qquad \text{... }(\textbf{\textit{ii}})$$

From equations (*i*) and (*ii*),

$W_1 = 3462$ N

$W_2 = 1538$ N

Wahl's stress factor for both the springs,

$$K_1 = K_2 = \frac{4C-1}{4C-4} + \frac{0.615}{C} = \frac{4\times 6-1}{4\times 6-4} + \frac{0.615}{6} = 1.2525$$

Maximum shear stress induced in the outer spring (τ_1),

$$850 = K_1 \times \frac{8\,W_1.C}{\pi(d_1)^2} \rightarrow d_1 = 8.83 \text{ say } 10 \text{ mm}$$

$D_1 = C.d_1 = 6\,d_1 = 6 \times 10 = 60$ mm

$$\frac{d_1}{d_2} = \frac{C}{C-2} \rightarrow d_2 = 5.88 \text{ say } 6 \text{ mm}$$

$D_2 = C.d_2 = 6 \times 6 = 36$ mm

Axial deflection for the outer spring (δ)

$$40 = \frac{8\,W_1.C^3.n_1}{G.d_1} \rightarrow n_1 = 5.35 \text{ say } 6$$

Assuming square and ground ends for the spring, the total number of turns of the outer spring,

$n_1' = 6 + 2 = 8$

Solid length of the outer spring,

$$L_{S1} = n_1'.\ d_1 = 8 \times 10 = 80 \text{ mm}$$

Since both the springs have the same solid length,

$$n_2'.d_2 = n_1'.d_1 \rightarrow n_2' = 13.3 \text{ say } 14$$

Also,

$$n_2' = n_2 + 2 \rightarrow n_2 = 12$$

Since both the springs have the same free length,

$$L_{F1} = L_{F2} = L_{S1} + \delta + 0.15\ \delta = 80 + 40 + 0.15 \times 40 = 126 \text{ mm}$$

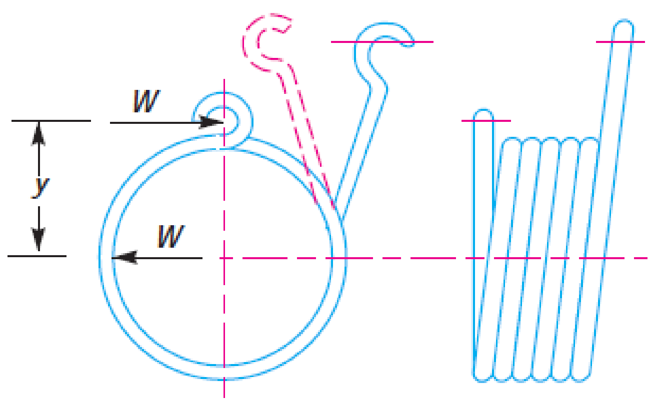
Outer diameter of the outer spring = $D_1 + d_1 = 60 + 10 = 70$ mm

Inner diameter of the outer spring = $D_1 - d_1 = 60 - 10 = 50$ mm

Outer diameter of the inner spring = $D_2 + d_2 = 36 + 6 = 42$ mm

Inner diameter of the inner spring = $D_2 - d_2 = 36 - 6 = 30$ mm

Helical Torsion Springs



The helical torsion springs, may be made from round, rectangular or square wire. These are wound in a similar manner as helical compression or tension springs but the ends are shaped to transmit torque. The primary stress in helical torsion springs is bending stress. The helical torsion springs are widely used for transmitting small torques as in door hinges, automobile starters etc. The radius of curvature of the coils changes when the twisting moment is applied to the spring. Thus, the wire is under pure bending.

According to A.M.Wahl, the bending stress in a helical torsion spring made of round wire is

$$\sigma_b = K \times \frac{32\,M}{\pi\, d^3}$$

where,

K = Wahl's stress factor
C = Spring index
M = Bending moment = $W \times y$
W = Load acting on the spring
y = Distance of load from the spring axis
d = Diameter of spring wire

Total angle of twist or angular deflection,

$$\theta = \frac{M.l}{E.I} = \frac{M \times \pi D.n}{E \times \pi\, d^4 / 64}$$

where,

l = Length of the wire = $\pi.D.n$
E = Young's modulus
I = Moment of inertia
D = Diameter of the spring
n = Number of active turns

When the spring is made of rectangular wire having width b and thickness t,

$$\sigma_b = K \times \frac{6\,M}{t.b^2}$$

Angular deflection,

$$\theta = \frac{12\,\pi M.D.n}{E.t.b^3}$$

Problem 7

A helical torsion spring of mean diameter 60 mm is made of a round wire of 6 mm diameter. If a torque of 6 N-m is applied on the spring, find the bending stress induced and the angular deflection of the spring in degrees. The spring index is 10 and modulus of elasticity for the spring material is 200 kN/mm^2. The number of effective turns may be taken as 5.5.

Given

D = 60 mm, d = 6 mm, M = 6 N-m, C = 10, E = 200 kN/mm^2, n = 5.5

To Find

σ_b, θ

Soln

Wahl's stress factor,

$$K = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} = 1.08$$

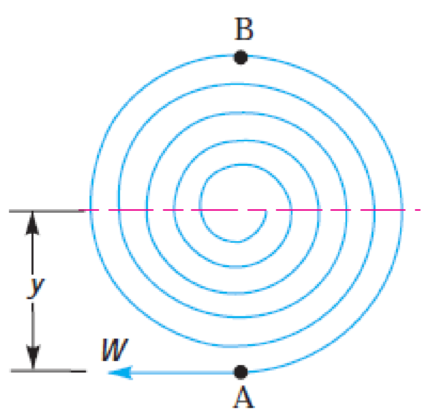
Bending stress induced,

$$\sigma_b = K \times \frac{32\, M}{\pi\, d^3} = 305.5 \text{ N/mm}^2$$

Angular deflection of the spring,

$$\theta = \frac{64\, M.D.n}{E.d^4} = 0.49 \text{ rad} = 0.49 \times \frac{180}{\pi} = 28°$$

Flat Spiral Spring



A flat spring is a long thin strip of elastic material wound like a spiral as shown in figure. These springs are frequently used in watches and gramophones etc. When the outer or inner end of this type of spring is wound up in such a way, the strain energy is stored into its spirals. This energy is utilized in any useful way while the spirals open out slowly. Usually the inner end of spring is clamped to an arbor while the outer end may be pinned or clamped. Since the radius of curvature of every spiral decreases when the spring is wound up, therefore the material of the spring is in a state of pure bending.

Let,

W = Force applied at the outer end A of the spring
y = Distance of centre of gravity of the spring from A
l = Length of strip forming the spring
b = Width of strip
t = Thickness of strip
I = Moment of inertia of the spring section = $b.t^3/12$
Z = Section modulus of the spring section = $b.t^2/6$

$$\ldots\left(\because Z = \frac{I}{y} = \frac{b.t^3}{12 \times t/2} = \frac{b.t^2}{6}\right)$$

When the end A of the spring is pulled up by a force W, then the bending moment on the spring, at a distance y from the line of action of W is given by

$$M = W \times y$$

The greatest bending moment occurs in the spring at B which is at a maximum distance from the application of W.

Bending moment at B,

$$M_B = M_{max} = W \times 2y = 2W.y = 2M$$

Maximum bending stress induced in the spring material,

$$\sigma_b = \frac{M_{max}}{Z} = \frac{2W \times y}{b.t^2/6} = \frac{12W.y}{b.t^2} = \frac{12M}{b.t^2}$$

Assuming that both ends of the spring are clamped, the angular deflection (in radians) of the spring is given by

$$\theta = \frac{M.l}{E.I} = \frac{12\ M.l}{E.b.t^3} \qquad \ldots\left(\because I = \frac{b.t^3}{12}\right)$$

Deflection, $\delta = \theta \times y$

Strain energy stored in the spring,

$$= \frac{1}{2} M.\theta$$

Problem 8

A spiral spring is made of a flat strip 6 mm wide and 0.25 mm thick. The length of the strip is 2.5 metre. Assuming the maximum stress of 800 MPa to occur at the point of greatest bending moment, calculate the bending moment, the number of turns to wind up the spring and the strain energy stored in the spring. Take E = 200 kN/mm^2.

Given

b = 6 mm, t = 0.25 mm, l = 2.5 m, σ_b = 800 MPa, E = 200 kN/mm^2

To Find

M, Number of turns, Energy stored in the spring

Soln

Maximum bending stress in the spring material (σ_b),

$$800 = \frac{12\ M}{b.t^2} \rightarrow M = 25 \text{ N-mm}$$

Angular deflection of the spring,

$$\theta = \frac{12\ M.l}{E.b.t^3} = 40 \text{ rad}$$

Since one turn of the spring is equal to 2π radians, therefore number of turns to wind up the spring

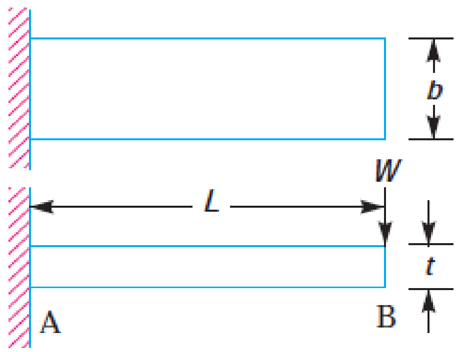
$$= 40 / 2\pi = 6.36 \text{ turns}$$

Strain energy stored in the spring,

$$= \frac{1}{2} M.\theta = 480 \text{ N-mm}$$

Leaf Springs

Leaf springs also known as flat springs are made out of flat plates. The advantage of leaf spring over helical spring is that the ends of the spring may be guided along a definite path as it deflects to act as a structural member in addition to energy absorbing device. Thus the leaf springs may carry lateral loads, brake torque, driving torque etc., in addition to shocks.



Consider a single plate fixed at one end and loaded at the other end as shown in figure. This plate may be used as a flat spring.

Let,

t = Thickness of plate
b = Width of plate
L = Length of plate or distance of the load W from the cantilever end.

Maximum bending moment at the cantilever end A,

$$M = W.L$$

Section modulus,

$$Z = \frac{I}{y} = \frac{b\,t^3/12}{t/2} = \frac{1}{6} \times b.t^2$$

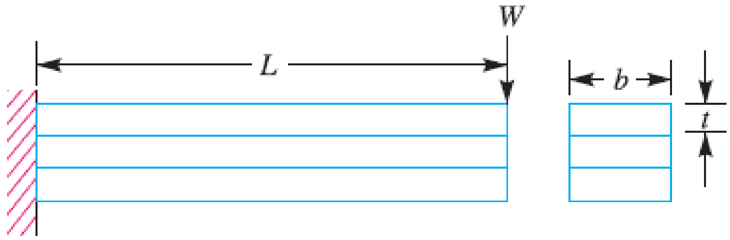
Bending stress in such a spring,

$$\sigma = \frac{M}{Z} = \frac{W.L}{\frac{1}{6} \times b.t^2} = \frac{6\,W.L}{b.t^2} \quad \text{... }(i)$$

Maximum deflection for a cantilever with concentrated load at the free end is given by

$$\delta = \frac{W.L^3}{3E.I} = \frac{W.L^3}{3E \times b.t^3/12} = \frac{4\,W.L^3}{E.b.t^3} \qquad \text{... } (ii)$$

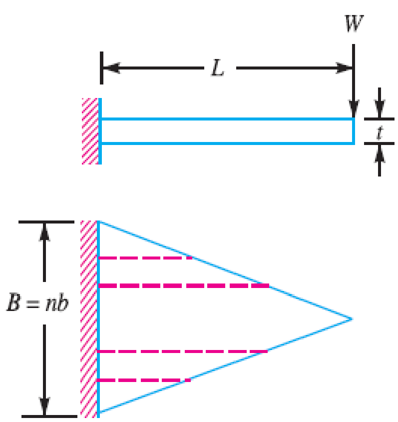
Spring such as automobile spring or semi-elliptical spring with length 2*L* and loaded in the centre by a load 2*W*, may be treated as a double cantilever. If the plate of cantilever is cut into a series of *n* strips of width *b* and these are placed as shown in figure, then equations (*i*) and (*ii*) may be written as



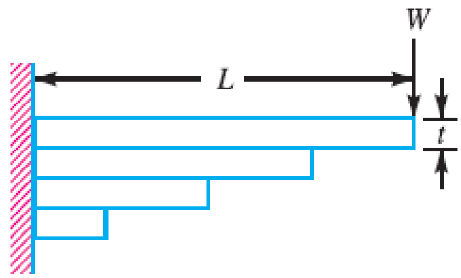
$$\sigma = \frac{6\,W.L}{n.b.t^2} \qquad \text{... } (iii)$$

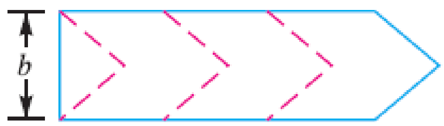
$$\delta = \frac{4\,W.L^3}{n.E.b.t^3} \qquad \text{... } (iv)$$

The above relations give the stress and deflection of a leaf spring of uniform cross-section. The stress at such a spring is maximum at the support. If a triangular plate is used as shown in figure, the stress will be uniform throughout.



If this triangular plate is cut into strips of uniform width and placed one below the other, as shown in figure to form a graduated or laminated leaf spring, then





$$\sigma = \frac{6\,W.L}{n.b.t^2} \qquad \dots (v)$$

$$\delta = \frac{6\,W.L^3}{n.E.b.t^3} \qquad \dots (vi)$$

where,

n = Number of graduated leaves

By the above arrangement, the spring becomes compact so that the space occupied by the spring is considerably reduced. When bending stress alone is considered, the graduated leaves may have zero width at the loaded end. But sufficient metal must be provided to support the shear. Therefore, it becomes necessary to have one or more leaves of uniform cross-section extending clear to the end. If the suffixes F and G are used to indicate the full length or uniform crosssection and graduated leaves, then

Bending stress for full length leaves,

$$\sigma_F = \frac{18\,W.L}{b.t^2\,(2\,n_G + 3\,n_F)}$$

Bending stress for graduated leaves,

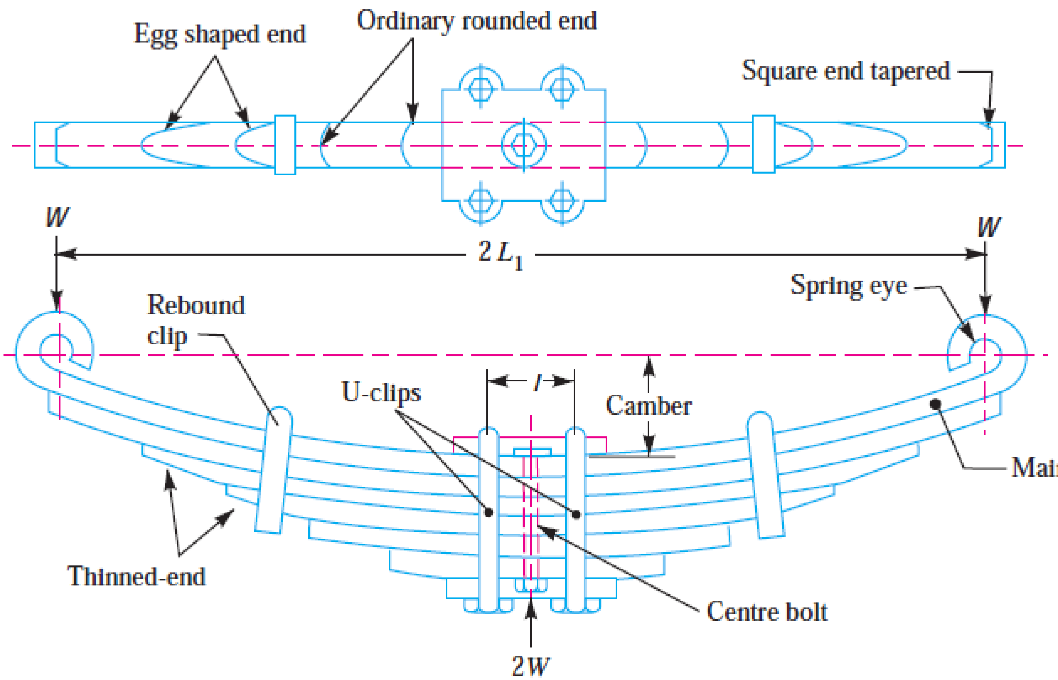
$$\sigma_G = \frac{12\, W.L}{b.t^2\ (2\, n_G + 3\, n_F)}$$

Deflection in full length and graduated leaves,

$$\delta = \frac{12\, W.L^3}{E.b.t^3\ (2\, n_G + 3\, n_F)}$$

Construction of Leaf Spring

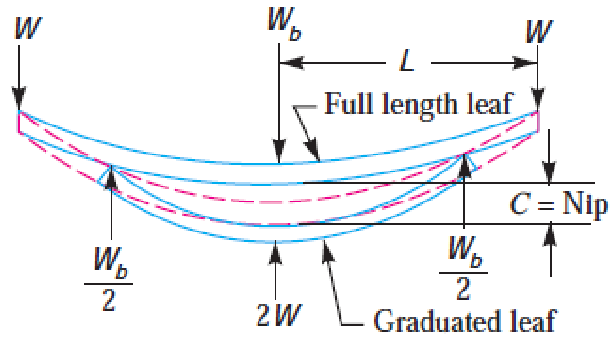
A leaf spring commonly used in automobiles is of semi-elliptical form as shown in figure.



Equalised Stress in Spring Leaves or Nipping

The stress in the full length leaves is 50% greater than the stress in the graduated leaves. In order to utilise the material to the best advantage, all the leaves should be equally stressed. This condition may be obtained in the following two ways:

By making the full length leaves of smaller thickness than the graduated leaves. In this way, the full length leaves will induce smaller bending stress due to small distance from the neutral axis to the edge of the leaf.



By giving a greater radius of curvature to the full length leaves than graduated leaves, as shown in figure, before the leaves are assembled to form a spring. By doing so, a gap or clearance will be left between the leaves. This initial gap, as shown by *C* in figure, is called nip. When the central bolt, holding the various leaves together, is tightened, the full length leaf will bend back as shown dotted in figure and have an initial stress in a direction opposite to that of the normal load. The graduated leaves will have an initial stress in the same direction as that of the normal load. When the load is gradually applied to the spring, the full length leaf is first relieved of this initial stress and then stressed in opposite direction. Consequently, the full length leaf will be stressed less than the graduated leaf. The initial gap between the leaves may be adjusted so that under maximum load condition the stress in all the leaves is equal, or if desired, the full length leaves may have the lower stress. This is desirable in automobile springs in which full length leaves are designed for lower stress because the full length leaves carry additional loads caused by the swaying of the car, twisting and in some cases due to driving the car through the rear springs. Let us now find the value of initial gap or nip *C*.

Consider that under maximum load conditions, the stress in all the leaves is equal. Then at maximum load, the total deflection of the graduated leaves will exceed the deflection of the full length leaves by an amount equal to the initial gap *C*.

$$\delta_G = \delta_F + C$$

$$C = \delta_G - \delta_F = \frac{6\, W_G \,.\, L^3}{n_G\, E.b.t^3} - \frac{4\, W_F.L^3}{n_F.E.b.t^3}$$

Length of Leaf Spring Leaves

Let,

$2L_1$ = Length of span or overall length of the spring

l = Width of band or distance between centres of *U*-bolts. It is the ineffective length of the spring

n_F = Number of full length leaves

n_G = Number of graduated leaves

n = Total number of leaves = $n_F + n_G$

Effective length of the spring,

$$2L = 2L_1 - l \qquad \text{... When band is used}$$

$$= 2L_1 - \frac{2}{3}l \qquad \text{... When } U\text{-bolts are used}$$

It may be noted that when there is only one full length leaf *i.e.* master leaf only, then the number of leaves to be cut will be n and when there are two full length leaves including one master leaf, then the number of leaves to be cut will be $(n - 1)$. If a leaf spring has two full length leaves, then the length of leaves is obtained as follows:

$$\text{Length of smallest leaf} = \frac{\text{Effective length}}{n-1} + \text{Ineffective length}$$

$$\text{Length of next leaf} = \frac{\text{Effective length}}{n-1} \times 2 + \text{Ineffective length}$$

Similarly, length of $(n - 1)$th leaf

$$= \frac{\text{Effective length}}{n-1} \times (n-1) + \text{Ineffective length}$$

The n^{th} leaf will be the master leaf and it is of full length. Since the master leaf has eyes on both sides, therefore

Length of master leaf = $2L_1 + \pi (d + t) \times 2$

where,

d = Inside diameter of eye

t = Thickness of master leaf

The approximate relation between the radius of curvature (R) and the camber (y) of the spring is given by

$$R = \frac{(L_1)^2}{2y}$$

The exact relation is given by

$$y\,(2R + y) = (L_1)^2$$

where,

L_1 = Half span of the spring

The maximum deflection (δ) of the spring is equal to camber (y) of the spring.

Problem 9

A semi-elliptical laminated vehicle spring to carry a load of 6000 N is to consist of seven leaves 65 mm wide, two of the leaves extending the full length of the spring. The spring is to be 1.1 m in length and attached to the axle by two U-bolts 80 mm apart. The bolts hold the central portion of the spring so rigidly that they may be considered equivalent to a band having a width equal to the distance between the bolts. Assume a design stress for spring material as 350 MPa. Determine:

- Thickness of leaves
- Deflection of spring
- Diameter of eye
- Length of leaves
- Radius to which leaves should be initially bent

The standard thicknesses of leaves are: 5, 6, 6.5, 7, 7.5, 8, 9, 10, 11 etc. in mm.

Given

$2W = 6000$ N $\rightarrow$ $W = 3000$ N, $n = 7$, $b = 65$ mm, $n_F = 2$, $2L_1 = 1.1$ m $\rightarrow$ $L_1 = 550$ mm, $l = 80$ mm, $\sigma = 350$ MPa

To Find

t, δ, d, Length of leaves, R

Soln

Let,

t = Thickness of leaves

Effective length of the spring,

$2L = 2L_1 - l = 1100 - 80 = 1020$ mm $\rightarrow L = 1020 / 2 = 510$ mm

Number of graduated leaves,

$n_G = n - n_F = 7 - 2 = 5$

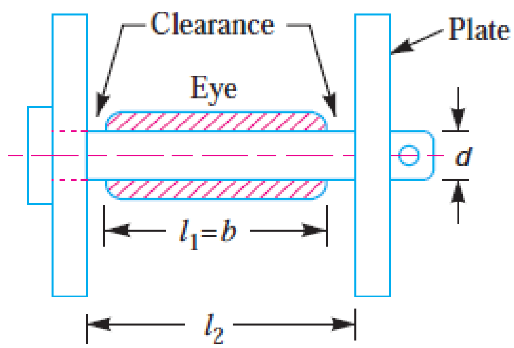
Assuming that the leaves are not initially stressed, the maximum stress (σ_F) for extra full length leaves,

$$350 = \frac{18\, W.L}{b.t^2\,(2n_G + 3n_F)} \rightarrow t = 8.7 \text{ say } 9 \text{ mm}$$

Deflection of spring,

$$\delta = \frac{12\, W.L^3}{E.b.t^3\,(2n_G + 3n_F)} = 30 \text{ mm}$$

The inner diameter of eye is obtained by considering the pin in the eye in bearing, because the inner diameter of the eye is equal to the diameter of the pin.



Let,

d = Inner diameter of the eye or diameter of the pin

l_1 = Length of the pin which is equal to the width of the eye or leaf

i.e, b = 65 mm

p_b = Bearing pressure on the pin which may be taken as 8 N/mm^2.

Load on pin (W),

$3000 = d \times l_1 \times p_b \rightarrow d = 5.77$ say 6 mm

Let us now consider the bending of the pin. Since there is a clearance of about 2 mm between the plate and eye as shown in figure, therefore length of the pin under bending,

$l_2 = l_1 + 2 \times 2 = 65 + 4 = 69$ mm

Maximum bending moment on the pin,

$$M = \frac{W \times l_2}{4} = 51\,750 \text{ N-mm}$$

Section modulus,

$$Z = \frac{\pi}{32} \times d^3 = 0.0982\, d^3$$

Bending stress (σ_b),

$$80 = \frac{M}{Z} \rightarrow d = 18.7 \text{ say } 20 \text{ mm}$$

Take the inner diameter of eye or diameter of pin (d) as 20 mm. Let us now check the pin for induced shear stress. Since the pin is in double shear, therefore load on the pin (W),

$$3000 = 2 \times \frac{\pi}{4} \times d^2 \times \tau \rightarrow \tau = 4.77 \text{ N/mm}^2\text{, which is safe.}$$

Ineffective length of the spring,

$l = 80$ mm ... U-bolts are considered equivalent to a band

$$\text{Length of the smallest leaf} = \frac{\text{Effective length}}{n-1} + \text{Ineffective length}$$

$$= \frac{1020}{7-1} + 80 = 250 \text{ mm}$$

$$\text{Length of the 2nd leaf} = \frac{1020}{7-1} \times 2 + 80 = 420 \text{ mm}$$

$$\text{Length of the 3rd leaf} = \frac{1020}{7-1} \times 3 + 80 = 590 \text{ mm}$$

$$\text{Length of the 4th leaf} = \frac{1020}{7-1} \times 4 + 80 = 760 \text{ mm}$$

$$\text{Length of the 5th leaf} = \frac{1020}{7-1} \times 5 + 80 = 930 \text{ mm}$$

$$\text{Length of the 6th leaf} = \frac{1020}{7-1} \times 6 + 80 = 1100 \text{ mm}$$

The 6^{th} and 7^{th} leaves are full length leaves and the 7^{th} leaf *i.e.* the top leaf will act as a master leaf.

Length of the master leaf,

$$= 2L_1 + \pi (d + t) 2 = 1282.2 \text{ mm}$$

Let,

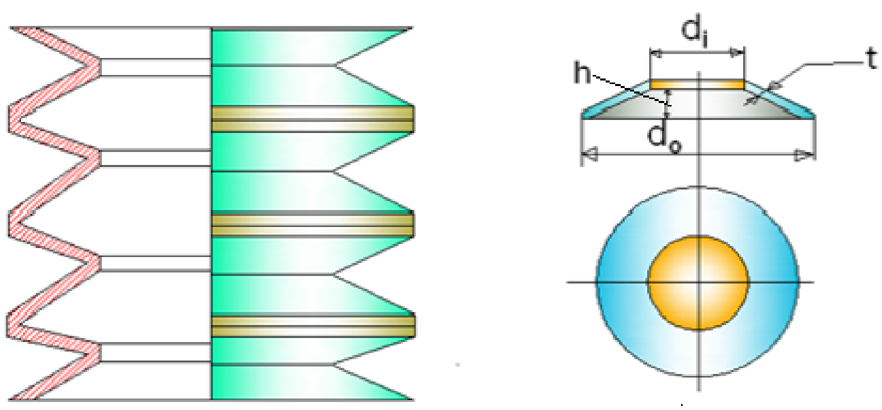
R = Radius to which the leaves should be initially bent

y = Camber of the spring

WKT,

$$y (2R - y) = (L_1)^2 \rightarrow R = 5056.5 \text{ mm} \qquad \ldots (y = \delta)$$

Belleville or Disc Spring



Disc spring, also called Belleville spring are used where high capacity compression springs must fit into small spaces. Each spring consists of several annular discs that are dished to a conical shape as shown in figure. They are stacked up one on top of another. When the load is applied, the discs tend to flatten out, and this elastic deformation constitutes the spring action. In safety valve the disc springs are used.

The relation between the load "F" and the axial deflection "y" of each disc,

$$F = \frac{4Ey}{(1 - v^2)Md_o^2}\left[(h - y)(h - \frac{y}{2})t + t^3 \right]$$

Maximum stress induced at the inner edge,

$$\sigma_i = \frac{4Ey}{(1 - v^2)Md_o^2}\left[C_1 (h - \frac{y}{2}) + C_2 t\right]$$

Maximum stress at the outer edge,

$$\sigma_o = \frac{4Ey}{(1 - v^2)Md_o^2}\left[C_1 (h - \frac{y}{2}) - C_2 t\right]$$

Where

C_1 and C_2 are constraints
t = thickness
h = height of spring
M = constant
F = axial force
d_o= outer diameter
d_i = inner diameter
υ = Poisson's ratio
E = Modulus of elasticity
y = axial deflection

Problem 9

A disc spring of outer diameter 156 mm and inner diameter 60 mm is dished by 4 mm. The maximum permissible deflection of the spring is 50% of the free height of the spring. The spring is made of the alloy steel for which allowable compressive stress is 720 N/mm^2, modulus of elasticity is 210×10^3 N/mm^2 , and Poisson's ratio is 0.3. Determine:

- ❖ Thickness of the spring

❖ Axial load carrying capacity of the disc spring

Given

d_o = 156 mm, d_i = 60mm, h = 4 mm, y = 0.5h = 2 mm, σ = 720 N/mm^2, E = 210 $\times 10^3$ N/mm^2, υ = 0.3

To Find

t, F

Soln

Maximum stress induced at the inner edge,

$$\sigma_i = \frac{4Ey}{(1-v^2)Md_o^2}\left[C_1\left(h-\frac{y}{2}\right) + C_2 t\right] \rightarrow t = 2 \text{ mm}$$

Axial load,

$$F = \frac{4Ey}{(1-v^2)Md_o^2}\left[(h-y)\left(h-\frac{y}{2}\right)t + t^3\right] = 1996.3 \text{ N}$$