TOPIC : 6 – EQUATIONS REDUCIBLE TO STANDARD TYPES

Type - V

Form the equation of the type of

$$f(x^m p, y^n q, z) = 0$$

case: (i) $m \neq 1, n \neq 1$ then $X = x^{1-m}$, $Y = y^{1-n}$

case: (ii) $m = n = 1$ then put $X = \log x$

$Y = \log y$

Next we follow type (3)

Type: 6

Eqn of the type $f(z^m p, z^m q) = 0 \rightarrow (1)$

& $f_1(x, z^m p) = f_2(y, z^m q) \rightarrow (2)$

case: (i) if $m \neq -1$ put $Z = z^{m+1} \Rightarrow$ Type (1) $\Rightarrow$ Type (4)

case: (ii) if $m = -1$ then put $Z = \log z$

$\Rightarrow$ Type (1) $\Rightarrow$ Type (4)

(c) Solve: $p^2+x^2y^2q^2=x^2z^2$

Sol:

$$p^2+x^2y^2q^2=x^2z^2$$

$\div x^2$, $$\frac{p^2}{x^2}+y^2q^2=z^2$$

$$x^{-2}p^2+y^2q^2=z^2$$

$$(x^{-1}p)^2+(yq)^2=z^2 \rightarrow (1)$$

This is of the form $f(x^m p, y^n q, z)=0$

Here $m=-1$, $n=1$

put $X=x^{1-m}$ $\quad Y=y^{1-n}\log y$

$X=x^{1+1}$ $\quad Y=y^{1-1}\log y$

$X=x^2$ $\quad Y=\log y$

$\frac{\partial X}{\partial x}=2x$ $\quad Q=\frac{\partial z}{\partial Y}$

$p=\frac{\partial z}{\partial x}$ $\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$ $\quad q=Q\frac{1}{y}$

$\quad yq=Q$

$p=2x\cdot P$

$\frac{p}{2x}=P$

$x^{-1}p=2P$

$(1) \Rightarrow (2P)^2+Q^2=z^2 \rightarrow (2)$

This is of the form $f(P,Q,z)=0$

we use Type (3)

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$P = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$Q = a\frac{dz}{du}$$

$$(2) \Rightarrow \left(2\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$(4+a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{4+a^2}}$$

$$\frac{dz}{z} = \frac{1}{\sqrt{4+a^2}}\, du$$

$$\int \frac{dz}{z} = \int \frac{1}{\sqrt{4+a^2}}\, du$$

$$\log z = \frac{1}{\sqrt{4+a^2}}\, u + b$$

$$\log z = \frac{1}{\sqrt{4+a^2}}\,(x+ay) + b$$

$$\log z = \frac{1}{\sqrt{4+a^2}}\,(x^2 + a\log y) + b$$

2. Solve: $x^2p^2+y^2q^2=z^2$

Sol:

$$x^2p^2+y^2q^2=z^2$$

$$(xp)^2+(yq)^2=z^2 \rightarrow ①$$

This eqn is of the form $f(x^m p, y^n q, z)=0$

Here $m=1$, $n=1$.

Put $X=\log x$ $\quad Y=\log y$

$$\frac{\partial X}{\partial x}=\frac{1}{x} \qquad \frac{\partial Y}{\partial y}=\frac{1}{y}$$

$$P=\frac{\partial z}{\partial X} \qquad Q=\frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$p=P\cdot\frac{1}{x} \qquad q=Q\cdot\frac{1}{y}$$

$$xp=P \qquad yq=Q$$

Sub in eqn ①, we get

$$P^2+Q^2=z^2 \rightarrow ②$$

This eqn is of the form $f(P,Q,z)=0$

Let $u=X+aY$

$$\frac{\partial u}{\partial X}=1 \qquad \frac{\partial u}{\partial Y}=a$$

$$P=\frac{\partial z}{\partial X}=\frac{dz}{du}\cdot\frac{\partial u}{\partial X} \qquad Q=\frac{\partial z}{\partial Y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial Y}$$

$$P=\frac{dz}{du} \qquad Q=a\frac{dz}{du}$$

Sub in ⑤ we get

$$\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2=z^2 \rightarrow ③$$

$$\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2=z^2$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2=z^2$$

$$\left(\frac{dz}{du}\right)^2=\frac{z^2}{1+a^2}$$

$$\frac{dz}{du}=\frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z}\,dz=\frac{1}{\sqrt{1+a^2}}\,du$$

$$\int\frac{dz}{z}=\int\frac{1}{\sqrt{1+a^2}}\,du$$

$$\log z=\frac{1}{\sqrt{1+a^2}}u+c$$

$$\log z=\frac{1}{\sqrt{1+a^2}}(x+ay)+c$$

$$\log z=\frac{1}{\sqrt{1+a^2}}(\log x+a\log y)+c$$

which is the complete solution.

① Solve $z^2(p^2+q^2) = x^2+y^2$

Sol: Given $z^2(p^2+q^2) = x^2+y^2$

$$(zp)^2+(zq)^2 = x^2+y^2 \rightarrow ①$$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^m q)$$

Here $m \neq -1$,

Put $Z = z^{m+1}$

$$\Rightarrow Z = z^{1+1} = z^2.$$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$P = 2zp$$

$$\frac{P}{2} = zp$$

Similarly, $\frac{Q}{2} = zq$

Sub in eqn ①, we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2+y^2$$

$$P^2+Q^2 = 4(x^2+y^2)$$

$$P^2 - 4x^2 = -Q^2 + 4y^2.$$

This eqn is of the form $f_1(x,p) = f_2(y,Q)$

$\therefore\ p^2 - 4x^2 = 4y^2 - Q^2 = 4a^2$

$p^2 = 4a^2 + 4x^2 \qquad Q^2 = -4a^2 + 4y^2$

$p = 2\sqrt{a^2+x^2} \qquad Q = 2\sqrt{y^2-a^2}$

$$dz = P\,dx + Q\,dy$$

$$dz = 2\sqrt{a^2+x^2}\,dx + 2\sqrt{y^2-a^2}\,dy$$

$$\int dz = 2\int\sqrt{a^2+x^2}\,dx + 2\int\sqrt{y^2-a^2}\,dy$$

$$z = 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$

$$z = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right) + b$$

$$= x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\left[\sinh^{-1}\left(\frac{x}{a}\right) - \cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$

2. Solve : $p^2 + q^2 = z^2(x^2+y^2)$

Sol: $p^2+q^2 = z^2(x^2+y^2) \rightarrow (1)$

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2+y^2$$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^n q)$$

Here $m = -1$. put $Z = \log z$.

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$P = \frac{1}{z}\cdot p$$

Similarly, $Q = \frac{1}{z}\cdot q$

Sub in eqn (1), we get

$$P^2 - x^2 = y^2 - Q^2$$

This eqn is of the form

$$f_1(x, P) = f_2(y, Q) \quad \text{Type (4)}$$

$$P^2 - x^2 = y^2 - Q^2 = a^2$$

$$P^2 - x^2 = a^2 \qquad y^2 - Q^2 = a^2$$

$$P^2 = x^2 + a^2 \qquad Q^2 = y^2 - a^2$$

$$P = \sqrt{x^2 + a^2} \qquad Q = \sqrt{y^2 - a^2}$$

$$dz = P\,dx + Q\,dy$$

$$dz = \sqrt{x^2 + a^2}\, dx + \sqrt{y^2 - a^2}\, dy$$

$$\int dz = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{y^2 - a^2}\, dy$$

$$z = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 + x^2}$$

$$+ \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b.$$

$(\because z = \log z$

$$\log z = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{2}\right) + \frac{x}{2} \sqrt{a^2 + x^2}$$

$$+ \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b.$$