(ii) $a|b \Rightarrow b = q_1 a$ and $a|c \Rightarrow c = q_2 a,$ $, for\ some\ integers\ q_1, q_2$

$\therefore\ \alpha b + \beta c = \alpha(q_1 a) + \beta(q_2 a)$

$= (\alpha q_1)a + (\beta q_2)a$

$= (\alpha q_1 + \beta q_2)a, \alpha q_1 + \beta q_2$, is an integer.

$\Rightarrow a|(\alpha b + \beta c).$

(iii) $a|b \Rightarrow \quad b = q_1 a$

$\therefore bc = (q_1 a)c = q_1(ac) = q_1(ca) = (q_1 c)a$

$\Rightarrow a|bc, \forall\ b \in Z.$

Note:

The expression $\alpha b + \beta c$ is called a linear combination of $b\ and\ c$. Thus, by part 2, if a is a factor of $b\ and\ c$, then a is also a factor of any linear combination of $b\ and\ c$. In particular, $a|(b+c)\ and\ a|(b-c)$.

The floor function can be used to determine the number of positive integers less than or equal to a positive integer a and divisible by a positive integer b, as the next theorem shows.

THEOREM 4: Let a and b be any positive integers. Then the number of positive integers $\leq a$ and divisible by b is $[{}^{a}/_{b}]$.

Proof:

Suppose there are k positive integers $\leq a$ and divisible by b.

we need to show that $k = [{}^{a}/_{b}]$.

The positive multiples of b less than or equal to a are $b, 2b, \dots, kb$.

Clearly, $kb \leq a, i.e. k \leq {}^{a}/_{b}$.

Further, $(k+1)b > a$. Thus, $k + 1 > {}^{a}/_{b}\ or\ {}^{a}/_{b} - 1 < k$.

$\therefore {}^{a}/_{b} - 1 < k \leq {}^{a}/_{b}$.

Thus, k is the largest integer less than or equal to ${}^{a}/_{b}$, so $k = [{}^{a}/_{b}]$.

Hence the proof.

For example, the number of positive integers *≤ 2076* and divisible by *19* is *[2076/19]=[109.26316] =109.*

Union, Intersection and Complement

Let A be a finite set and $|A|$ the number of elements in A .

For example, if $\mathrm{A} = \{3, 5, 8, 17\}$, then $|A| = 4$.

Let $A\ and\ B$ be any two sets. Their union $A \cup B$ is the set of elements belonging to $A\ or\ B$; their intersection $A \cap B$ consists of the common elements ; A' denotes the complement of A , that is , the set of elements in the universal set that are not in A.

Principle of Inclusion and Exclusion:

Let A and B be finite sets. Let $|A \cup B| = |A| + |B| - |A \cap B|$

Likewise, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

THEOREM 6: (The Inclusion-Exclusion Principle)

Let $A_1, A_2, \dots, A_n$ be n finite sets. Then $|\cup_{i=1}^{n} A_i| = \sum_{1\le i\le n}|A_i| - \sum_{1\le i\le j\le n}|A_i \cap A_j| + \sum_{1\le i\le j\le k\le n}|A_i \cap A_j \cap A_k| - \cdots . + (-1)^{n+1}|\cap_{i=1}^{n} A_i|$.

Example 3: Find the number of positive integers ≤ 2076 and divisible by neither 4 nor 5.

Solution:

Let $A = \{x \in N: x \le 2076\ and\ divisible\ by\ 4\}$ and

$B = \{x \in N: x \le 2076\ and\ divisible\ by\ 5\}$. Then

$|A \cup B| = |A| + |B| - |A \cap B|$

$= \left\lfloor \frac{2076}{4} \right\rfloor + \left\lfloor \frac{2076}{5} \right\rfloor - \left\lfloor \frac{2076}{20} \right\rfloor$

$= 519 + 415 - 103 = 831.$

Thus, among the first 2076 positive integers, there are 2076-831 = 1245 integers not divisible by 4 or 5.

Example 4: Find the number of positive integers ≤ 3000 and divisible by 3, 5 or 7.

Solution:

Let $A, B\ and\ C$ denote the sets of positive integers ≤ 3000 and divisible by 3, 5 or 7.

By the inclusion-exclusion principle.

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

$= \left\lfloor \frac{3000}{3} \right\rfloor + \left\lfloor \frac{3000}{5} \right\rfloor + \left\lfloor \frac{3000}{7} \right\rfloor - \left\lfloor \frac{3000}{15} \right\rfloor - \left\lfloor \frac{3000}{35} \right\rfloor - \left\lfloor \frac{3000}{21} \right\rfloor + \left\lfloor \frac{3000}{105} \right\rfloor$

=1000+600+428-200-85-142+28

=1629.