DISCRETE MATHEMATICS

UNIT I LOGIC & PROOF

PART-A

1. **Express the statement, "The crop will be destroyed if there is a flood," in symbolic form.**
   Solution:
   Let C: The crop will be destroyed.
   F: There is a flood
   Symbolic form: $F \rightarrow C$
2. **State the truth table of "If tigers have wings then the earth travels round the sun."**
   Solution: Let P: Tigers have wings. "F"
   Q: The earth travels round the sun. "F"
   Therefore, given statement is $P \rightarrow Q$, has the truth value "T".

P	Q	$P \rightarrow Q$
F	F	T

3. **Give the converse and contrapositive of the implication " If it is raining, then I get wet".**
   Solution: P: It is raining.
   Q: I get wet.
   Converse: $(Q \rightarrow P)$ If I get wet, then it is raining.
   Contrapositive: $(\neg Q \rightarrow \neg P)$ If I do not get wet, then it is not raining.
4. **What are the contrapositive, the converse and the inverse of the conditional statement. "If you work hard then you will be rewarded."**
   Solution:
   P: You work hard — $\neg P$: You will not work hard
   Q: You will be rewarded — $\neg Q$: You will not be rewarded.
   Converse: $(Q \rightarrow P)$ You will be rewarded only if you work hard.
   Contrapositive: $(\neg Q \rightarrow \neg P)$ If you will not be rewarded then you will not work hard.
   Inverse: $(\neg P \rightarrow \neg Q)$ If you will not work hard then you will not be rewarded.
5. **Define Tautology with an example.**
   Solution: A statement that is true for all possible values of propositional variables is called a tautology or universally valid formula or a logical truth.
   Example: $P \vee \neg P$ is a tautology.
6. **Using truth table, show that the proposition $p \vee \neg(p \wedge q)$ is a tautology.**
   Solution:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Therefore, $p \vee \neg(p \wedge q)$ is a tautology.

7. **Show that the proposition P→Q and ⌉P∨Q are logically equivalent.**

Solution:

We should prove that $(P\to Q) \Leftrightarrow \urcorner P\vee Q$.

i.e., To prove $(P\to Q) \leftrightarrow \urcorner P\vee Q$ is tautology.

P	Q	P→Q	⌉P	⌉P∨Q	(P→Q) ↔⌉P∨Q
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(P\to Q) \Leftrightarrow \urcorner P\vee Q$.

8. **State the rules of inference theory.**

Solution:

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation of s is tautologically implied by any one or more of the preceding formulas in the derivation.

Rule CP: If we can derive S from R and a set of premises, then we can derive R→S from the set of premises alone.

9. **Determine whether the conculsion C follows logically from the premises H_1 and H_2 or not. H_1: P→Q, H_2:P, C:Q.**

Solution:

P	Q	P→Q	(P→Q)∧P	[(P→Q)∧P]→Q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Therefore, conclusion is valid.

10. **Write the negation of the statement (∃x) (∀y) p(x,y).**

Solution:

Given : $(\exists x)\ (\forall y)\ p(x,y)$

Negation : $(\forall x)\ (\exists y)\ \overline{p(x,y)}$.

11. **Write the scope of the quantifiers in the formula (x) (P(x)→(∃y)R(x, y))**

Solution:

The scope of (x) is P(x)→(∃y)R(x,y).

The scope of (∃y) is R(x,y).

12. **"Every parrot is ugly"-Express using quantifiers.**

Solution: P(x): x is a parrot;

Q(x): x is ugly.

Symbolic form: (x)(P(x)→Q(x)).

13. **Symbolize the statement "Some men are genius"**

Solution:

Let M(x): x is a man

G(x): x is genius.

Symbolic form: $(\exists x)(M(x) \wedge G(x))$.

14. **Symbolize the statement "All men are giants."**

Solution:

Case(i): Using G(x): x is a giant.

M(x): x is a man.

The given statement can be symbolized as $(\forall x)[(M(x) \rightarrow G(x)]$

Case(ii): However, if we restrict the variable x to the universe which is the class of men, then the statement is $(\forall x)G(x)$.

15. **Symbolize the statement "x is the father of the mother of y".**

Solution:

P(x): x is a person

F(x,y): x is father of y.

M(x,y): x is mother of y.

Symbolic form: $(\exists z)[P(z) \wedge F(x,z) \wedge M(z,x)]$.

16. **Explain the two types of quantifiers through example.**

Solution:

(i)Universal Quantifiers: The quantifier "all" is called the universal Quantifier and we shall denote it by $\forall x$.

Eg: For all x, x is an integer is written as $(\forall x)\ I(x)$.

(ii)Existential Quantifiers: The quantifier "some" is called the existential quantifier and we shall denote it by $\exists x$.

Eg: There exists an x such that x is a man is written as $(\exists x)\ M(x)$.

PART-B

Problems based on Logical Equivalence:

1. **Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.**
2. **Without using the truth table, prove that $\neg p \rightarrow (q \rightarrow r) \Leftrightarrow (q \rightarrow (p \vee r)$.**
3. **Show that $\neg(p \wedge \neg(p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent.**
4. **Show that $(\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$**

Problems based on Normal forms:

5. **Without using truth table find the PCNF and PDNF of $P \rightarrow (Q \wedge P) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$.**
[Ans: PCNF: $(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$
PDNF: $(P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$].
6. **Obtain the product of sum canonical form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$**
7. **Find the PCNF of $(P \vee R) \wedge (P \vee \neg Q)$. Also find its PDNF, without using truth table.**
8. **Obtain the PCNF and PDNF of $(P \wedge Q) \vee (\neg P \wedge R)$**

<u>Problems based on Inference theory of Statement Calculus:</u>

9. **Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$.**
10. **Show that $(P \to Q) \wedge (R \to S)$, $(Q \wedge M) \wedge (S \to N)$, $\neg(M \wedge N)$ and $(P \to R) \Rightarrow \neg P$**
11. **Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$, $\neg M$.**
12. **Prove that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.**
13. **Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ & Q**
14. **Using conditional proof prove that $\neg P \vee Q$, $\neg Q \vee R$, $R \to S \Rightarrow P \to S$**
15. **Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R)$ and $(Q \to S)$.**
16. **Show that the following set of premises are inconsistent**
    (i) If Jack misses many classes through illness, then he fails high school
    (ii) If Jack fails high school, then he is uneducated
    (iii) If Jack reads a lot of books, then he is not uneducated.
    (iv) Jack misses many classes through illness and reads a lot of books.
17. **Show that the following implication by using indirect method. $(R \to \neg Q)$, $R \vee S$, $S \to \neg Q$, $P \to Q \Rightarrow \neg P$.**
18. **Using Indirect method of proof, derive $p \to \neg s$ from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p.**

<u>Problems based on Predicate Calculus:</u>

19. **Prove that $(\forall x)\,(P(x) \to Q(x))$, $(\forall x)\,(R(x) \to \neg Q(x)) \Rightarrow (\forall x)\,(R(x) \to \neg P(x))$.**
20. **Show that the conclusion $(\forall x)\,(F(x) \to \neg S(x))$ follows from the premises $(\exists x)(F(x) \wedge S(x)) \to (y)\,(M(y) \to W(y))$ and $(\exists y)\,(M(y) \wedge \neg W(y))$.**
21. **Show that $(\forall x)\,(P(x) \vee Q(x)) \Rightarrow (\forall x)\,(P(x) \vee (\exists x)\,Q(x))$ by indirect method of proof.**
22. **Show that $(x)\,(P(x) \to Q(x) \wedge (x)\,(Q(x) \to R(x)) \Rightarrow (x)\,(P(x) \to R(x)$**

UNIT II-COMBINATORICS

PART-A

23. Use mathematical induction to show that $n! \geq 2^{n-1}, n = 1, 2, 3$ ....

Solution:

Let $p(n): n! \geq 2^{n-1}, n = 1,2,3$ ...

Step 1: **To prove p(1) is true.**

$1! \geq 2^{1-1}$

$1! \geq 2^0$

$1 \geq 1 \Rightarrow$ p(1) is true.

Step 2: Assume that p(k) is true.

$k! \geq 2^{k-1}, k = 1,2, \ldots.$

Step 3: To prove p(k+1) is true.

i.e. $(k+1)! \geq 2^{(k+1)-1}$

$(k+1)! = (k+1)(k!)$

$\geq (k+1)2^{k-1}$

$\geq 2\ 2^{k-1}$ [Since (k+1)≥2]

$\geq 2^{(k+1)-1}$

Hence p(k+1) is true.

24. State the pigeon hole principle.

Solution:

If n pigeons are assigned to m pigeon holes, and m<n, then at least one pigeonhole contains two or more pigeons.

25. How many students must be in a class to guarantee that atleast two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points.

Solution:

There are 101 possible scores 0,1,2………100

By Pigeonhole principle, we have assuming 102 students there must be atleast 2 students with the same score.

Therefore 102 students must be in a class to guarantee that atleast two students reveive the same score.

26. If seven colours are used to paint 50 bicycles, then show that atleast 8 bicycles will be the same colour.

Solution:

Here, Number of Pigeon = n = Number of bicycle = 50

Number of Holes = m = Number of colours = 7

By generalized Pigeon hole principle, we get

$$\left[\frac{n-1}{m}\right] + 1 = \left[\frac{50-1}{7}\right] + 1 = 8$$

Therefore, Atleast 8 bicycles will have the same colour.

<u>Problems based on Mathematical Induction:</u>

27. Prove, by mathematical induction, that for all $n \geq 1$, n^3+2n is a multiple of 3.

28. Prove by the principle of mathematical induction for a positive integer,

$$1^2 + 2^2 + 3^2 + \dots\dots\dots\dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

29. Use mathematical induction to show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots\dots\dots\dots\dots\dots + \frac{1}{\sqrt{n}} > \sqrt{n}, n \geq 2.$

30. Use the mathematical induction to prove the inequality $n<2^n$ for all positive integer n.

31. Use mathematical induction to show that $1+2+2^2+\dots\dots\dots\dots\dots 2^n=2^{n+1}-1$ for all non negative integers n.

32. Using mathematical induction prove that

$$1^2 + 3^2 + 5^2 + \dots\dots\dots\dots\dots\dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$