Predicate Calculus

Def: Predicates

Predicates are simple statements which turn out to be propositions involving the variables whose values are not well specified. Every predicate tells something about one (or) more objects.

Eg: The statement "x is greater than 4" has two parts. The first part, the variable x, is the subject of the statement. The second part the predicate "is greater than 4" – refers to a property that the subjects of the statement.

Notation:

We can denote the statement "x is greater than 4" by p(x), here p denote the predicate "is greater than 4" and x is the variable.

Simple statement function:

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. Eg: If "x is a teacher" is denoted by T(x). ~~Compound Stat~~ it is a statement function. If x is replaced by John, then "John is a teacher" is a statement.

Compound Statement function:

A Compound Statement function is obtained by combining one (or) more simple statement functions by logical connectives.

Eg: $A(x)$: x is an apple
$r(x)$: x is red.

compound Statement function Such as
$A(x) \wedge r(x)$, $A(x) \rightarrow r(x)$, $A(x) \vee \neg r(x)$ etc....

Quantifiers:

Certain declarative sentences involve words that indicate quantity such as "all", some, none (or) one' These words help determine the answer to the question "How many?" Since such words indicate quantity they are called quantifiers.

Eg: (1) Not all vegetarians are healthy persons.
(2) Some birds cannot fly.

Quantification:

When all the variables in a statement function are assign values, the resulting statement has a truth value. There is another important way called Quantification to Create a statement ~~function~~ from a statement function.

Types of Quantifiers:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier :

The quantifier "all" is called the universal quantifier and we shall denote it by $\forall x$.

Eg: For all x, x is an integer is written as
$(\forall x)\, I(x)$ (or) $(x)\, I(x)$.

Note :

The symbol $(\forall x)$ represents each of the following phrases, having the same meaning as 'all'
1. (for all x) 2. (for every x) 3. (for each x)
4. (everything x is such that)
5. (eachthing x is such that).

Existential Quantifier :

The quantifier "some" is called the existential quantifier and we shall denote it by $\exists x$.

Eg: There exists an x such that x is a man is written as $(\exists x)\, M(x)$.

Note :

The symbol $(\exists x)$ represents each of the following phrases, having the same meaning as 'some'
(for some x), (for x such that),
(there exists an x such that)
(there is an x such that)
(there is atleast one x such that)

Nested (more than one) Quantifiers :

When we consider propositional func. containing two (or) more variables it is possible, quantifiers occurs in combinations with respect to the variable. Nested quantifiers are quantifiers that occur within the scope of other quantifiers. For Eg: $(\forall x)(\exists y)\,(x+y=0)$

Free and Bound Variable:

Given a formula containing a part of the form $(x)\,P(x)$ (or) $(\exists x)\,P(x)$, Such a part is called an x-bound part of the formula. Any occurence of x in an x-bound part of a formula is Called a bound occurence of x, while any occurrence of x (or) of any variable that is not a bound occurence is called free occurence.

Further, the formula $P(x)$ either in $(x)\,P(x)$ (or) in $(\exists x)\,P(x)$ is described as the Scope of the quantifier.

Consider the following formulas

1) $(x)\,P(x,y)$
2) $(x)(P(x) \rightarrow Q(x))$
3) $(x)(P(x) \rightarrow (\exists y)\,R(x,y))$
4) $(x)(P(x) \rightarrow R(x)) \vee (x)(P(x) \rightarrow Q(x))$
5) $(\exists x)\,[P(x) \wedge Q(x)]$
6) $(\exists x)\,[P(x)] \wedge Q(x)$;

In (1) $P(x,y)$ is the Scope of quantifier and both occurences of x are bound occurences, while the occurence of y is free occurence.

In (2) the Scope of the universal quantifier is $P(x) \rightarrow Q(x)$ and all occurrences of x are bound.

In (3), the scope of $(\forall x)$ is $P(x) \rightarrow (\exists y) R(x,y)$ while the scope of $(\exists y)$ is $R(x,y)$. All occurences of both x & y are bound occurence.

In (4), the scope of 1st quantifier is $P(x) \rightarrow R(x)$" and the scope of 2nd is $P(x) \rightarrow Q(x)$. All occurence of x are bound occurence.

In (5), the scope of $(\exists x)$ is $P(x) \wedge Q(x)$

In (6), the scope of $(\exists x)$ is $P(x)$ and the last occurence of x in $Q(x)$ is free.

Problems

1) Let $P(x)$: x is a person

$F(x,y)$: x is father of y.

$M(x,y)$: x is the mother of y

Write the predicate "x is ~~fat~~ the father of the mother of y"

Soln:

In order to symbolize the predicate we name a person called z as the mother of y. Obviously we want to say that x is the father of z and z the mother of y

It is assumed that such a person z exists

We symbolize the predicate as

$$(\exists z)[P(z) \wedge F(x,z) \wedge M(z,y)]$$

2) Symbolize the expression
"All the world likes a mother"

Soln:

The above statement states that everybody likes a mother.

Now let $p(x)$: x is a person
$M(x)$: x is a mother.
$R(x,y)$: x likes y.

The required expression is

$$(x)\left[P(x) \rightarrow (y)\left(p(y) \wedge M(y) \longrightarrow R(x,y)\right)\right]$$

3) Give the symbolic form of the statement "Every book with a blue cover is a mathematics book"

Soln:

Let $B(x)$: x is a book with blue cover.
$M(x)$: x is a mathematics book.

Symbolic form: $(x)\left(B(x) \rightarrow M(x)\right)$

4) So symbolize the statement "Some men are genius"

Soln:

Let $M(x)$: x is a man
$G(x)$: x is genius

Symbolic form: $(\exists x)\left(M(x) \wedge G(x)\right)$

Universe of Discourse:

The variables which are quantified stand for only those objects which are members of a particular set (or) class. Such a restricted class is called the universe of discourse (or) the domain of individuals (or) simply the universe.

Eg:

Symbolize the statement "All men are giants".

Solu:

Case (i) Using $G(x)$: x is a giant
$M(x)$: x is a man

the given statement can be symbolized as $(\forall x)(M(x) \rightarrow G(x))$.

Case (ii): However, if we restrict the variable x to the universe which is the class of men, then the statement is $(\forall x)\, G(x)$.

Consider the statement "Given any positive integer there is a greater positive integer". Symbolize this statement with and without using the set of positive integers as the universe of discourse.

Solu:

Case (i). Let the variable x & y be restricted to the set of positive integers. Then the above statement can be expressed as follows.

For all x, there exists a y such that y is greater than x. If $G(x,y)$ is "x is greater than y". then the given statement is $(\forall x)(\exists y)\, G(y,x)$

Case (ii): If we do not impose the restriction on the universe of discourse and if we write $p(x)$ for "x is a positive integer" then we can symbolize the given statement is

$$(x)\left[p(x) \rightarrow (\exists y)\left(p(y) \wedge G(y,x)\right)\right]$$

3) Express the statement "Every student in this class has studied calculus" as a universe quantification.

Soln:

Let $p(x) = x$ has studied calculus.

Case (i): Taking the universe as the set of all students in this class.

$\therefore$ Symbolic form $(\forall x)\, p(x)$

Case (ii): Taking the universe as the set of all students in the college and using the predicate

$s(x)$: x is in this class

Symbolic form: $(\forall x)\left(s(x) \rightarrow p(x)\right)$