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Tautology:

A Statement that is true for all possible values of its propositional variables is called a Tautology (or) universally valid formula (or) a logical truth.

Eg: $p \vee \neg p$ is always a tautology.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Contradiction

A Statement that is always false for the truth values of the components is called a Contradiction.

Eg: $p \wedge \neg p$ is always a Contradiction.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

1) Using truth table show that the proposition $p \vee \neg(p \wedge q)$ is a tautology.

Soln:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since all truth values of a given formula are true, hence given formula is a tautology.

2) Prove that $(\neg q \wedge p) \wedge q$ is a contradiction.

p	q	$\neg q$	$\neg q \wedge p$	$(\neg q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	F	F	F

Since all the truth values of given formula are all false, the given formula is a contradiction.

3) Prove that $(\neg p \to q) \to (q \to p)$ is neither a tautology nor a contradiction.

Soln:

p	q	$(\neg p \to q)$	$q \to p$	$(\neg p \to q) \to (q \to p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	T

Since the last column is neither all false nor true therefore the given formula is neither a tautology nor a contradiction.

Contrapositive:

If $p \to q$ is a conditional statement, then $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

Note:

The conditional statment $p \to q$ and its contrapositive $\neg q \to \neg p$ have same truth values.

Truth table

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

1) Give the converse and the contrapositive of the implication "If it is raining, then I get wet".

Soln:

P : It is raining

Q : I get wet.

Converse ($Q \rightarrow P$): If I get wet, then it is raining.

Contrapositive: ($\neg Q \rightarrow \neg P$)

If I donot get wet, then it is not raining.

2) What are the contrapositive, the converse, and the inverse of the conditional statement "If you work hard then you will be rewarded."

Soln:

p : You work hard

Q : You will be rewarded.

Converse: $Q \rightarrow p$

You will be rewarded only if you work hard.

Contrapositive: $\neg Q \rightarrow \neg p$

If you will not be rewarded then you will not work hard.

Inverse: $\neg p \rightarrow \neg Q$.

If you will not work hard then you will not be rewarded.

Logical Equivalence:

Let p and q be two statement formulas, p is said to be logically equivalent to q if p and q have the same set of truth values or equivalently p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

(or)

The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

Notation: (i) $p \Leftrightarrow q$ (ii) $p \equiv q$

Ex 1. Show that $p \rightarrow Q$ & $\neg p \vee Q$ are logically equivalent (ie) $(p \rightarrow Q \equiv \neg p \vee Q)$

Proof: We should prove that $(p \rightarrow Q) \Leftrightarrow \neg p \vee Q$.

(ie) To prove $(p \rightarrow Q) \Leftrightarrow \neg p \vee Q$ is a tautology

p	Q	$p \rightarrow Q$	$\neg p$	$\neg p \vee Q$	$(p \rightarrow Q) \leftrightarrow \neg p \vee Q$
T	T	T	F	T	T
T	F	F	F	F	T
F	F	T	T	T	T
F	F	T	T	T	T

$\therefore (p \rightarrow Q) \Leftrightarrow \neg p \vee Q$.

2. When do you say that two compound propositions are equivalent?

Soln: The two compound propositions are said to be equivalent if they have identical truth values.

3) prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Proof:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

from truth table we see that $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$, have ~~r~~ same truth values

Hence $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Demorgan's laws

State:
(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
(2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proof:

(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

(2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Laws of logic

S.NO	Equivalence	Name
1.	$p \vee q \Leftrightarrow q \vee p$; $p \wedge q \Leftrightarrow q \wedge p$	Commutative law
2.	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative laws
3.	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive laws
4.	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	Demorgan's laws
5.	$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	Idempotent laws
6.	$p \wedge (p \vee q) \Leftrightarrow p$ $p \vee (p \wedge q) \Leftrightarrow p$	Absorption laws
7.	$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$	Identity laws
8.	$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$	Domination laws
9.	$\neg(\neg p) \Leftrightarrow p$	Double negation law
10.	$p \vee \neg p \Leftrightarrow T$ (or) $\neg p \vee p \Leftrightarrow T$ $p \wedge \neg p \Leftrightarrow F$ (or) $\neg p \wedge p \Leftrightarrow F$	Negation laws

1. Show that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$.

Soln:

$(P \to Q) \wedge (R \to Q)$	Reasons.
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$	Since $P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg P \wedge \neg R) \vee Q$	Distributive law
$\Leftrightarrow \neg (P \vee R) \vee Q$	Demorgan's law
$\Leftrightarrow (P \vee R) \to Q$	Since $\neg P \vee Q \Leftrightarrow P \to Q$.

2. Show that $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$

Soln:

$P \to (Q \to R)$	Reasons.
$\Leftrightarrow P \to (\neg Q \vee R)$	Since $Q \to R \Leftrightarrow \neg Q \vee R$
$\Leftrightarrow \neg P \vee (\neg Q \vee R)$	Since $P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg P \vee \neg Q) \vee R$	Associative law
$\Leftrightarrow \neg (P \wedge Q) \vee R$	De Morgan's law
$\Leftrightarrow (P \wedge Q) \to R$	Since $\neg P \vee Q \Leftrightarrow P \to Q$

3. Without using truth table show that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$

Soln:

LHS: $P \to (Q \to P)$	Reason.
$\Leftrightarrow \neg P \vee (\neg Q \vee P)$	$\because P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg Q \vee P) \vee \neg P$	Commutative law
$\Leftrightarrow \neg Q \vee (P \vee \neg P)$	Associative law
$\Leftrightarrow \neg Q \vee T$	$P \vee \neg P \Leftrightarrow T$
$\Leftrightarrow T$ —— (1)	$P \vee T \Leftrightarrow T$ —— (1)

RHS: $\neg p \to (p \to Q)$ — Reason

$\Leftrightarrow \neg\neg p \vee (\neg p \vee Q)$ — $\because p \to Q \Leftrightarrow \neg p \vee Q$

$\Leftrightarrow p \vee (\neg p \vee Q)$ — Double negation law

$\Leftrightarrow (p \vee \neg p) \vee Q$ — Associative law

$\Leftrightarrow T \vee Q$ — $p \vee \neg p \Leftrightarrow T$

$\Leftrightarrow T$ —— ② $\because p \vee T = T$.

From ① & ② we get

$$p \to (Q \to p) \Leftrightarrow \neg p \to (p \to Q)$$

4. Show that $(\neg p \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R) \Leftrightarrow R$

Soln:

$(\neg p \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R)$

$\Leftrightarrow ((\neg p \wedge \neg Q) \wedge R) \vee ((Q \vee p) \wedge R)$ — (by associative & distributive laws)

$\Leftrightarrow (\neg (p \vee Q) \wedge R) \vee ((p \vee Q) \wedge R)$ — [by Demorgan's & commutative laws]

$\Leftrightarrow [\neg (p \vee Q) \vee (p \vee Q)] \wedge R$ — (by distributive laws)

$\Leftrightarrow [\neg A \vee A] \wedge R$, where $A = p \vee Q$

$\Leftrightarrow T \wedge R$

$\Leftrightarrow R$

$\therefore (\neg p \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R) \Leftrightarrow R$.

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Tautological implications

A statement P is said said to tautologically imply a statement Q if and only if $P \to Q$ is a tautology. In this case, we write $P \Rightarrow Q$, read as "P implies Q"

(ie) ($P \Rightarrow Q$ i.e. P Tautologically implies Q) iff $P \to Q$ is tautology.

If $P \Rightarrow Q$ then P is called antecedent and Q is called consequent.

1) prove that $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$

Proof:

Let $S : (P \to Q) \wedge (Q \to R) \to (P \to R)$

To prove : S is a tautology.

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	S
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

$\therefore (P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$

Note: The above implication is called as hypothetical syllogism (or) Chain rule (or) law of syllogism.