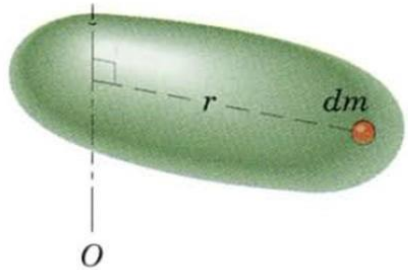
Mass of Inertia

Mass Moments of Inertia (*I*): Important in Rigid Body Dynamics

- *I* is a measure of distribution of mass of a rigid body w.r.t. the axis in question (constant property for that axis).
- Units are (mass)(length)^2 € kg.m^2



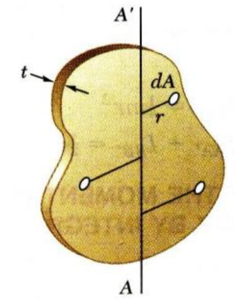
Consider a three dimensional body of mass *m*
Mass moment of inertia of this body about axis *O*-*O*:

$$I = \int r^2 \, dm$$

Integration is over the entire body.

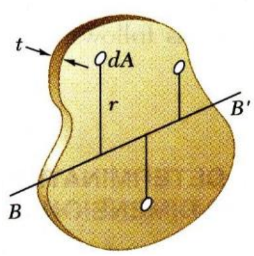
r = perpendicular distance of the mass element *dm* from the axis *O*-*O*

Moments of Inertia of Thin Plates



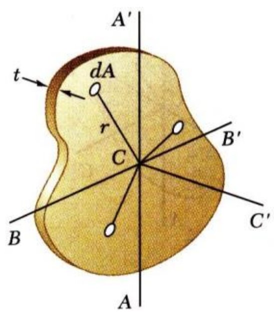
- For a thin plate of uniform thickness t and homogeneous material of density ρ, the mass moment of inertia with respect to axis AA' contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t \, I_{AA',area}$$



- Similarly, for perpendicular axis BB' which is also contained in the plate,

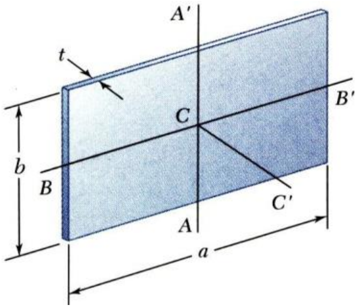
$$I_{BB'} = \rho t \, I_{BB',area}$$



- For the axis CC' which is perpendicular to the plate,

$$I_{CC'} = \rho t \, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area} \right)$$
$$= I_{AA'} + I_{BB'}$$

Moments of Inertia of Thin Plates

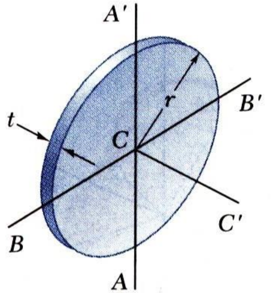


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t\, I_{AA',area} = \rho t \left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t\, I_{BB',area} = \rho t \left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

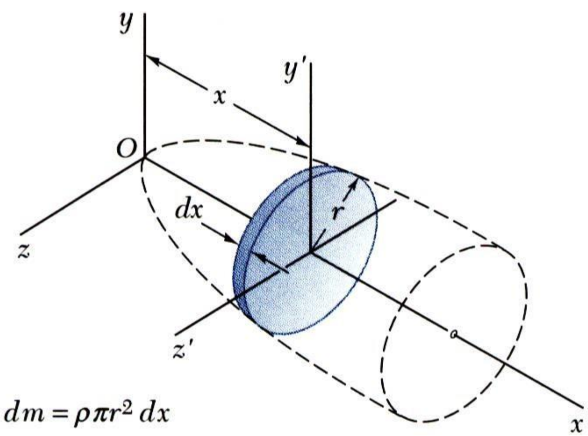
$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m \left(a^2 + b^2\right)$$



- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t\, I_{AA',area} = \rho t \left(\tfrac{1}{4} \pi r^4\right) = \tfrac{1}{4} m r^2$$

Moments of Inertia of a 3D Body by Integration



$$dm = \rho \pi r^2 \, dx$$

$$dI_x = \frac{1}{2} r^2 \, dm$$

$$dI_y = dI_{y'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

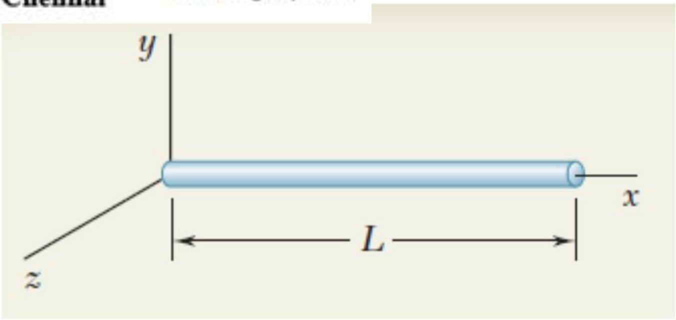
- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

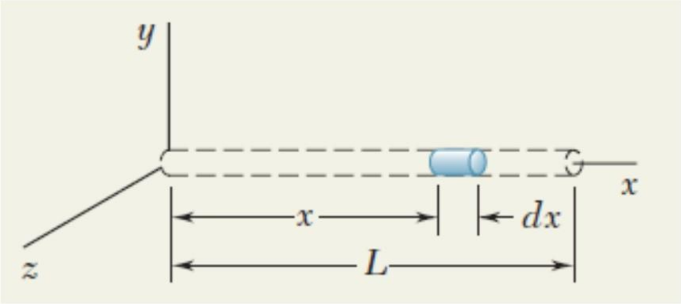
- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.

- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

Q. No. Determine the moment of inertia of a slender rod of length L and mass m with respect to an axis which is perpendicular to the rod and passes through one end of the rod.



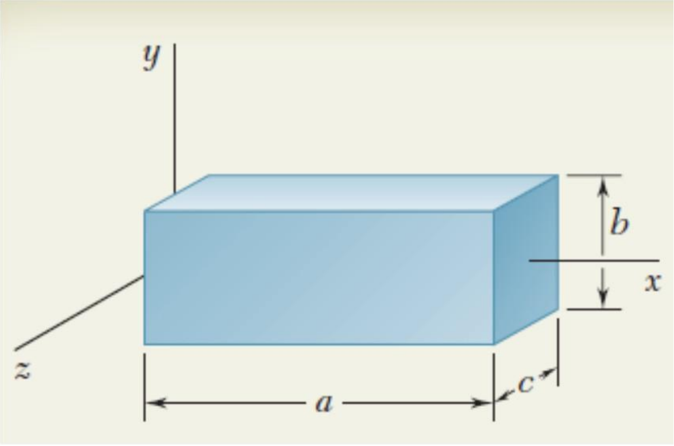
Solution

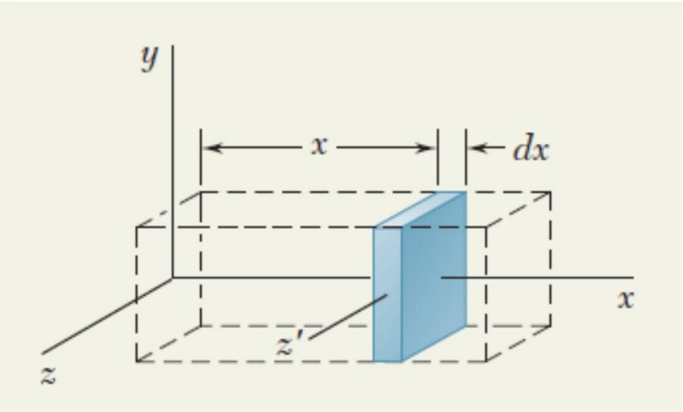


$$dm = \frac{m}{L}dx$$

$$I_y = \int x^2\,dm = \int_0^L x^2 \frac{m}{L}dx = \left[\frac{m}{L}\frac{x^3}{3}\right]_0^L \qquad I_y = \tfrac{1}{3}mL^2$$

Q. No. For the homogeneous rectangular prism shown, determine the moment of inertia with respect to z axis





Solution

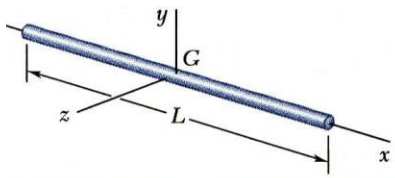
$$dm = \rho bc\,dx \qquad dI_{z'} = \tfrac{1}{12}b^2\,dm$$

$$dI_z = dI_{z'} + x^2\,dm = \tfrac{1}{12}b^2\,dm + x^2\,dm = (\tfrac{1}{12}b^2 + x^2)\rho bc\,dx$$

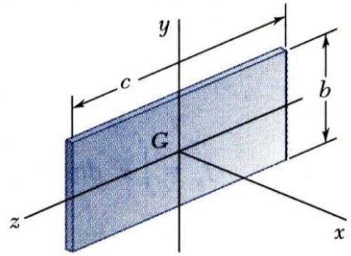
$$I_z = \int dI_z = \int_0^a (\tfrac{1}{12}b^2 + x^2)\rho bc\,dx = \rho abc(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2)$$

$$I_z = m(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2) \qquad I_z = \tfrac{1}{12}m(4a^2 + b^2)$$

MI of some common geometric shapes



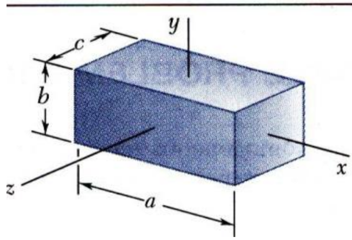
$I_y = I_z = \frac{1}{12} mL^2$



$I_x = \frac{1}{12} m(b^2 + c^2)$

$I_y = \frac{1}{12} mc^2$

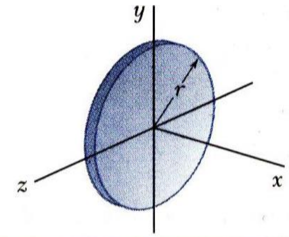
$I_z = \frac{1}{12} mb^2$



$I_x = \frac{1}{12} m(b^2 + c^2)$

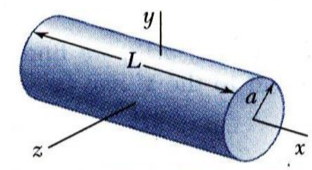
$I_y = \frac{1}{12} m(c^2 + a^2)$

$I_z = \frac{1}{12} m(a^2 + b^2)$



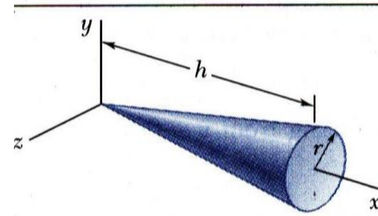
$I_x = \frac{1}{2} mr^2$

$I_y = I_z = \frac{1}{4} mr^2$



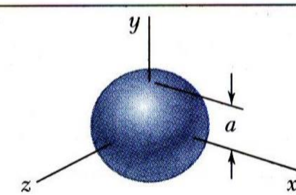
$I_x = \frac{1}{2} ma^2$

$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$



$I_x = \frac{3}{10} ma^2$

$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$



$I_x = I_y = I_z = \frac{2}{5} ma^2$