Define function of class A.

Solution : A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A.

◆ **Important Result**

(1) $L[1] = \frac{1}{s}$ where $s > 0$

(2) $L[t^n] = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2, \ldots$

(3) $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ where n is not a integer.

(4) $L[e^{at}] = \frac{1}{s-a}$ where $s > a$ or $s - a > 0$

(5) $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

(6) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s > 0$

(7) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s > 0$

(8) $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2 > a^2$

(10) $L[af(t) \pm bg(t)] = a\,L[f(t)] \pm b\,L[g(t)]$ [Linearity property]

Note : (1) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$

$e^{\infty} = 1 + \frac{\infty}{\lfloor 1} + \frac{\infty^2}{\lfloor 2} + \ldots$

(2) $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

(3) $\Gamma_{n+1} = n!$

(4) $\Gamma_{n+1} = \int_0^\infty x^n e^{-x}\, dx$

(5) $\Gamma_{n+1} = n\,\Gamma_n$

(6) $\Gamma_{1/2} = \sqrt{\pi}$

(7) $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

(8) $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

(9) $\sin^3\theta = \dfrac{1}{4}[3 \sin\theta - \sin 3\theta]$

(10) $\cos^3\theta = \dfrac{1}{4}[\cos 3\theta + 3 \cos\theta]$

(11) $\sin A \cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$

(12) $\cos A \sin B = \dfrac{1}{2}[\sin(A+B) - \sin(A-B)]$

(13) $\cos A \cos B = \dfrac{1}{2}[\cos(A+B) + \cos(A-B)]$

(14) $\sin A \sin B = -\dfrac{1}{2}[\cos(A+B) - \cos(A-B)]$

5.2 TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Result (1) : Prove that $L[1] = \dfrac{1}{s}$ where $s > 0$

Proof : We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

Here $f(t) = 1$

$$\therefore L[1] = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}[0 - 1] \text{ by note (2)}$$

$$= \frac{1}{s},\ s > 0$$

Result (2) : Prove that $L[t^n] = \frac{n!}{s^{n+1}}$ [n = 0, 1, 2, ...]

Proof : We know that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt = \int_0^\infty t^n\, d\left[\frac{e^{-st}}{-s}\right]$$

$$= t^n \left(\frac{e^{-st}}{-s}\right)\Bigg]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n\, t^{n-1}\, dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\, dt$$

i.e., $$L[t^n] = \frac{n}{s} L[t^{n-1}]$$

Similarly $$L[t^{n-1}] = \frac{n-1}{s} L[t^{n-2}]$$

$$L[t^{n-2}] = \frac{n-2}{s} L[t^{n-3}]$$

....

....

$$L[t^{n-(n-1)}] = \frac{n-(n-1)}{s} L\left[t^{[n-(n-1)]-1}\right]$$

$$= \frac{1}{s} L\left[t^0\right] = \frac{1}{s} L[1] = \frac{1}{s}\frac{1}{s}$$

$$\therefore L[t^n] = \frac{n}{s}\frac{n-1}{s}\cdots\frac{2}{s}\frac{1}{s}\frac{1}{s} = \frac{n!}{s^n}\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}} \text{ where } [n = 0, 1, 2, ...]$$

Result (3) Prove that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$ where n is not a integer.

Proof : We know that $$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt$$

Put $st = x$ as $t \to 0 \Rightarrow x \to 0$

$s\,dt = dx$ as $t \to \infty \Rightarrow x \to \infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$$

i.e., $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$ $\qquad [\because \int_0^\infty x^n e^{-x}\, dx = \Gamma_{n+1}]$

when n is a positive integer.

we get $\Gamma_{n+1} = n\,!$

$$L\left[t^n\right] = \frac{n\,!}{s^{n+1}}$$

II. PROBLEMS BASED ON TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Example 1 Find L[t]

Solution : $L[t^n] = \dfrac{n\,!}{s^{n+1}}$ [we know that]

$$L[t] = \frac{1\,!}{s^{1+1}} = \frac{1}{s^2}$$

Example 2 Find L [t³]

Solution : We know that $L[t^n] = \dfrac{n\,!}{s^{n+1}}$

$$L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Example 3 Find L[√t]

Solution : We know that $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$

$$L[\sqrt{t}] = L[t^{1/2}] = \frac{\Gamma_{1/2+1}}{s^{1/2+1}}$$

$$= \frac{\frac{1}{2}\Gamma_{1/2}}{s^{3/2}} \qquad [\because \Gamma_{n+1} = n\Gamma_n \; ; \; \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{\Gamma_{1/2}}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Example 4. Find L $[t^{3/2}]$

Solution :

We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[t^{3/2}\right] = \frac{\Gamma_{3/2+1}}{s^{3/2}+1} = \frac{\frac{3}{2}\Gamma_{3/2}}{s^{5/2}}$$

$$= \frac{\frac{3}{2}\Gamma_{1/2+1}}{s^{5/2}} = \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma_{1/2}}{s^{5/2}}$$

$$= \frac{\left(\frac{3}{4}\right)\sqrt{\pi}}{s^{5/2}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}}$$

Example 5.2.5. Find L $\left[\frac{1}{\sqrt{t}}\right]$

Solution : We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right] = \frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}$$

$$= \frac{\Gamma_{1/2}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

Result 4. Prove that L $[e^{at}] = \frac{1}{s-a}$ where $s > a$.

Proof : We know that

$$L[f(t)] = \int_C^{\infty} e^{-st} f(t)\, dt$$

$$L\left[e^{at}\right] = \int_0^\infty e^{-st}\ e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}[0 - 1] = \frac{1}{s-a} \text{ where } s-a > 0$$

Example 6. Find the value $L\left[e^{3t}\right]$

Solution : We know that

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

Example 7 Find $L[e^{3t+5}]$

Solution :

W.K.T $L[e^{at}] = \dfrac{1}{s-a}$

$$L[e^{3t+5}] = L[e^{3t}\ e^{5}]$$

$$= e^5\ L\ [e^{3t}] = e^5\left[\frac{1}{s-3}\right] = \frac{e^5}{s-3}$$

Example 8 Find $L\left[\dfrac{e^{at}}{a}\right]$

Solution : W.K.T $L[e^{at}] = \dfrac{1}{s-a}$

$$L\left[\frac{e^{at}}{a}\right] = \frac{1}{a}L\left[e^{at}\right] = \frac{1}{a}\left[\frac{1}{s-a}\right]$$

Example 9 Find $L[2^t]$

Solution: W.K.T. $L[e^{at}] = \dfrac{1}{s-a}$

$$L[2^t] = L\left[e^{\log 2^t}\right]$$

$$= L\left[e^{t\log 2}\right]$$

$$= L\left[e^{(\log 2)t}\right]$$

$$= \frac{1}{s-\log 2}$$

Result 5. Prove that $L[e^{-at}] = \frac{1}{s+a}$, $(s+a) > 0$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= -\frac{1}{s+a}[0 - 1]$$

$$= \frac{1}{s+a} \text{ where } (s+a) > 0$$

Example 10. Find $L[e^{-bt}]$

Solution : W.K.T $L[e^{-at}] = \frac{1}{s+a}$

$$L[e^{-bt}] = \frac{1}{s+b}$$

Example 11. Find $L[2e^{-3t}]$

Solution : W.K.T. $L[e^{-at}] = \frac{1}{s+a}$

$$L[2e^{-3t}] = 2L[e^{-3t}]$$

$$= 2\left[\frac{1}{s+3}\right] = \left[\frac{2}{s+3}\right]$$

Result 6. Prove that $L[\sin at] = \frac{a}{s^2+a^2}$ $(s > 0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\int e^{ax} b(x)\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s \sin at - a \cos at]\right]_0^\infty \text{ by Note 7.}$$

$$= 0 - \left[\frac{(-a)}{s^2 + a^2}\right] = \frac{a}{s^2 + a^2} \text{ where } s > 0.$$

Example 5.2.12. Find L [sin 2t]

Solution : W.K.T $L[\sin at] = \dfrac{a}{a^2 + a^2}$

$$L[\sin 2t] = \frac{2}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 4}$$

Example 5.2.13. Find L [sin π t]

Solution : W.K.T $L[\sin at] = \dfrac{a}{s^2 + a^2}$

$$L[\sin \pi t] = \frac{\pi}{s^2 + \pi^2}$$

Result : 7. Prove that $L[\cos at] = \dfrac{s}{s^2 + a^2}$ $(s > 0)$

Proof : W.K.T. $L[f(t) = \int_0^\infty e^{-st} f(t)\, dt$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}[-s \cos at + a \sin at]\right]_0^\infty$$

$$= 0 - \left[\frac{1}{s^2 + a^2}(-s)\right]$$

$$= \frac{s}{s^2 + a^2} \quad (s > 0)$$

Example 5.2.14. Find L [cos 2t]

Solution : W.K.T. $L[\cos at] = \dfrac{s}{s^2 + a^2}$

$$L[\cos 2t] = \frac{s}{s^2 + 4}$$

Example 15 Prove that $L[\cos at] = \frac{s}{s^2+a^2}$ **and** $L[\sin at] = \frac{a}{s^2+a^2}$

Solution : By Euler's theorem

$$e^{ix} = \cos x + i \sin x$$

$$e^{iat} = \cos at + i \sin at$$

$$L[e^{iat}] = L[\cos at + i \sin at]$$

$$= L[\cos at] + i\, L[\sin at]$$

$$L[\cos at] + i\, L[\sin at] = L[e^{iat}]$$

$$= \frac{1}{s - ia}$$

$$= \left[\frac{1}{s-ia}\right] \left[\frac{s+ia}{s+ia}\right]$$

$$= \frac{s+ia}{s^2+a^2}$$

Equating real & Imaginary parts we get

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

Example 16 Find $L[\cos(at+b)]$

Solution : $L[\cos(at+b)]$

$$= L[\cos at \cos b - \sin at \sin b]$$

$$= \cos b\, L[\cos at] - \sin b\, L[\sin at]$$

$$= \cos b \left[\frac{s}{s^2+a^2}\right] - \sin b \left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{s \cos b - a \sin b}{s^2+a^2}$$

Example 17 Find $L[\sin^2 2t]$

Solution : $L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right] = \frac{1}{2} L[1 - \cos 4t]$

$$= \frac{1}{2}\left[L[1] - L[\cos 4t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

Example 18 Find L [sin 5t cos 2t]

Solution : $L[\sin 5t \cos 2t] = \frac{1}{2} L[\sin 7t + \sin 3t]$ by Note 11.

$$= \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

Example 19 Find L[(sin t − cos t)^2]

Solution : $L[(\sin t - \cos t)^2] = L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$

$$= L[1 - \sin 2t] = L[1] - L[\sin 2t]$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

Result 8. Prove that L[sinh *at*] = $\frac{a}{s^2 - a^2}$ where s > | a |

Proof : $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} - e^{-at}] = \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}, \; s > |a|$$

Result 9. Prove that L [cosh at] = $\frac{s}{s^2-a^2}$, s > | a |

Proof : $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} + e^{-at}] = \frac{1}{2}\left[L[e^{at}] + L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}, \; s > |a|$$

Result 11. Prove that $L[f'(t)] = s\,L[f(t)] - f(0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d[f(t)]$$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L[f(t)]$$

$$= s\,L[f(t)] - f(0)$$

Result 12. Prove that $L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d[f'(t)]$$

$$= e^{-st} f'(t)\Big]_0^\infty - \int_0^\infty f'(t)(-s)e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \varphi(s+a)$$

III. PROBLEMS BASED ON FIRST SHIFTING THEOREM AND SECOND SHIFTING THEOREM

Example **Find L [$t^n e^{-at}$]**

Solution :
$$L[t^n e^{-at}] = \left[L(t^n)\right]_{s \to (s+a)} = \left[\frac{n!}{s^{n+1}}\right]_{s \to (s+a)} = \frac{n!}{(s+a)^{n+1}}$$

Example **Find L [$e^{-at}\cos bt$]**

Solution :
$$L[e^{-at}\cos bt] = \left[L[\cos bt]\right]_{s \to (s+a)} = \left[\frac{s}{s^2+b^2}\right]_{s \to (s+a)} = \frac{s+a}{(s+a)^2+b^2}$$

Example **Find L [$e^{at}\sinh bt$]**

Solution :
$$L[e^{at}\sinh bt] = \left[L[\sinh bt]\right]_{s \to (s-a)} = \left[\frac{b}{s^2-b^2}\right]_{s \to (s-a)} = \frac{b}{(s-a)^2-b^2}$$

Example **Find L $\left[e^t t^{-1/2}\right]$**

Solution :
$$L\left[e^t t^{-1/2}\right] = \left[L[t^{-1/2}]\right]_{s \to (s-1)}$$

$$= \left[\frac{\Gamma -1/2+1}{s^{-1/2+1}}\right]_{s \to (s-1)} = \left[\frac{\Gamma 1/2}{s^{1/2}}\right]_{s \to (s-1)}$$

$$= \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s \to (s-1)} = \left[\sqrt{\frac{\pi}{s}}\right]_{s \to (s-1)}$$

$$\sqrt{\ \ \pi}$$

$$L[t f(t)] = -\varphi'(s)$$

Corollary :- If $L[f(t)] = \varphi(s)$ then $L[t^n f(t)] = (-1)^n \varphi^n(s)$.

Proof : W.K.T. $L[t f(t)] = -\varphi'(s)$

$$L[t^2 f(t)] = L[t \,.\, t f(t)]$$

$$= -\frac{d}{ds} L[t f(t)]$$

$$= -\frac{d}{ds}\left[\frac{-d}{ds} L f(t)\right]$$

$$= (-1)^2 \frac{d^2}{ds^2}[L f(t)]$$

$$= (-1)^2 \frac{d^2}{ds^2} \varphi(s)$$

....
....
....

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \varphi(s) = (-1)^n \varphi^n(s)$$

PROBLEMS BASED ON TRANSFORMS OF DERIVATIVES

Example 1. Find L [t sin 2t]

Solution : W.K.T. $L[t^n f(t)] = (-1)^n \varphi^n(s)$

$$L(t \sin 2t) = -\frac{d}{ds}[L(\sin 2t)] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left[\frac{-4s}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

Example 2. Find L $[t^2 e^{-3t}]$

Solution : W.K.T $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\varphi(s)]$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}] = \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{2}{(s+3)^3}$$

i.e., $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds$

PROBLEMS BASED ON INTEGRALS OF TRANSFORM

Example 8 **Find** $L\left[\frac{1-e^t}{t}\right]$

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L[1-e^t]\,ds = \int_s^\infty \left[\frac{1}{s}-\frac{1}{s-1}\right]ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty = \left[\log\frac{s}{s-1}\right]_s^\infty$$

$$= \left[\log\frac{s}{s(1-1/s)}\right]_s^\infty = \left[\log\frac{1}{1-1/s}\right]_s^\infty$$

$$= 0 - \log\frac{s}{s-1} = \log\left(\frac{s-1}{s}\right)$$

Example 9 **Find** $L\left[\frac{\sin at}{t}\right]$ **[A.U., March 1996]**

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left[\frac{s}{a}\right] = \tan^{-1}\left[\frac{a}{s}\right]$$

Note : $\cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$$\frac{s}{a} = \cot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{s}{a}$$

PROBLEMS BASED ON INITIAL VALUE AND FINAL VALUE THEOREM

Example 5.4.1. If $L[f(t)] = \frac{1}{s(s+a)}$, find $\underset{t\to\infty}{Lt} f(t)$ and $\underset{t\to 0}{Lt} f(t)$

Solution : $\underset{t\to 0}{Lt} f(t) = \underset{s\to\infty}{Lt}\, s\,.\,F(s)$

$$= \underset{s\to\infty}{Lt}\, s\frac{1}{s(s+a)} = \underset{s\to\infty}{Lt}\frac{1}{s+a} = \frac{1}{\infty} = 0$$

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\, s\,F(s) = \underset{s\to 0}{Lt}\, s\frac{1}{s(s+a)}$$

$$= \underset{s\to 0}{Lt}\frac{1}{s+a} = \frac{1}{a}$$

Example 2. Verify the initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution : Initial value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\text{L.H.S} = \lim_{t\to 0} f(t) = 1 + 1 = 2$$

$$\text{R.H.S} = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1 + 1 = 2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Initial value theorem verified.

TRANSFORM OF PERIODIC FUNCTIONS

Define periodic function and state its Laplace transform formula.

◆ Def. Periodic

A function $f(x)$ is said to be "periodic" if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function $f(t)$ with period p given by $\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

Put $t = u + p$ in the second integral

i.e., $u = t - p$ $\quad t \to p \Rightarrow u \to 0$

i.e., $du = dt$ $\quad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^p e^{-st} f(t)\, dt + \int_0^\infty e^{-(u+p)s} f(u+p)\, du$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-su} f(u)\, du \quad [\because f(u+p) = f(u)]$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$L[f(t)] = \int_0^p e^{-st} f(t)\, dt + e^{-sp} L[f(t)]$$

$$[1 - e^{-sp}] L[f(t)] = \int_0^p e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

Example 2 **Find the Laplace Transform of**

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \text{ with } f(t + 2a) = f(t)$$

Solution : $$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}\, (2a - t)\, dt \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a - t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{[1 - e^{-as}]^2}{s^2 (1 - e^{-as})(1 + e^{-as})} = \frac{1 - e^{-as}}{s^2 (1 + e^{-as})} = \frac{1}{s^2} \tanh\left[\frac{as}{2}\right]$$