TRANSFORM OF PERIODIC FUNCTIONS

Define periodic function and state its Laplace transform formula.

◆ Def. Periodic

A function $f(x)$ is said to be "periodic" if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function $f(t)$ with period p given by $\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt$$

Put $t = u + p$ in the second integral

i.e., $u = t - p$ $\quad t \to p \Rightarrow u \to 0$

i.e., $du = dt$ $\quad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-(u+p)s} f(u+p)\,du$$

$$= \int_0^p e^{-st} f(t)\,dt + e^{-sp} \int_0^\infty e^{-su} f(u)\,du \quad [\because f(u+p) = f(u)]$$

$$= \int_0^p e^{-st} f(t)\,dt + e^{-sp} \int_0^\infty e^{-st} f(t)\,dt \quad [\because u \text{ is a dummy variable}]$$

$$L[f(t)] = \int_0^p e^{-st} f(t)\,dt + e^{-sp} L[f(t)]$$

$$[1 - e^{-sp}]\,L[f(t)] = \int_0^p e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\,dt$$

Example 2 **Find the Laplace Transform of**

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \text{ with } f(t + 2a) = f(t)$$

Solution :

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st} (2a - t)\, dt \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left[-t\,\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a - t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1 - e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{[1 - e^{-as}]^2}{s^2 (1 - e^{-as})(1 + e^{-as})} = \frac{1 - e^{-as}}{s^2 (1 + e^{-as})} = \frac{1}{s^2} \tanh\left[\frac{as}{2}\right]$$