Def. Exponential order

A function $f(t)$ is said to be of exponential order if

$$\underset{t \to \infty}{\text{Lt}} \; e^{-st} f(t) = 0$$

Example 1 **Show that x^n is of exponential order as $x \to \infty$, $n > 0$.**

Solution :

$$\underset{x \to \infty}{\text{Lt}} \; e^{-ax} x^n = \underset{x \to \infty}{\text{Lt}} \frac{x^n}{e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{n x^{n-1}}{a e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right] \text{[Apply L' Hospital Rule]}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{n(n-1)\dots 1}{a^n e^{ax}} \text{ [Repeating this process we get]}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{n!}{a^n e^{ax}} \text{ [Applying L'Hospital's rule]}$$

$$= \frac{n!}{\infty} = 0$$

Hence x^n is of exponential order.

Example **Show that t^2 is of exponential order.**

Solution : $$\underset{t \to \infty}{\text{Lt}} \; e^{-st} t^2 = \underset{t \to \infty}{\text{Lt}} \frac{t^2}{e^{st}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

[Apply L'Hospital's rule]

$$= \underset{t \to \infty}{\text{Lt}} \frac{2t}{s e^{st}} \left[\frac{\infty}{\infty} \text{ form}\right] \text{[Apply L'Hospital's Rule]}$$

$$= \underset{t \to \infty}{\text{Lt}} \frac{2}{s^2 e^{st}} = \frac{2}{\infty}$$

$$= 0$$

Hence t^2 is of exponential order.

Example **Show that the function**

$f(t) = e^{t^2}$ is not of exponential order.

Solution : $$\underset{t \to \infty}{\text{Lt}} \; e^{-st} e^{t^2} = \underset{t \to \infty}{\text{Lt}} \; e^{-st + t^2}$$

$$= e^{\infty} = \infty$$

So $f(t) = e^{t^2}$ is not of exponential order.

Define function of class A.

Solution : A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A.

♦ Important Result

(1) $L[1] = \frac{1}{s}$ where $s > 0$

(2) $L[t^n] = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2, \ldots$

(3) $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ where n is not a integer.

(4) $L[e^{at}] = \frac{1}{s-a}$ where $s > a$ or $s - a > 0$

(5) $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

(6) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s > 0$

(7) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s > 0$

(8) $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2 > a^2$

(10) $L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]$ [Linearity property]

Note : (1) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$

$$e^{\infty} = 1 + \frac{\infty}{\lfloor 1} + \frac{\infty^2}{\lfloor 2} + \ldots$$

(2) $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

(3) $\Gamma_{n+1} = n!$

(4) $\Gamma_{n+1} = \int_0^\infty x^n e^{-x}\, dx$

(5) $\Gamma_{n+1} = n\,\Gamma_n$

(6) $\Gamma_{1/2} = \sqrt{\pi}$

(7) $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]$

(8) $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}[a \cos bx + b\sin bx]$

(9) $\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

(10) $\cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$

(11) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(12) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

(13) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(14) $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

5.2 TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Result (1) : Prove that $L[1] = \frac{1}{s}$ where $s > 0$

Proof : We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

Here $f(t) = 1$

$$\therefore L[1] = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}[0 - 1] \text{ by note (2)}$$

$$= \frac{1}{s},\ s > 0$$