UNIT 4 – ANALYTIC FUNCTIONS

Let $z = x+iy$ be a complex variable

Then $w = f(z) = u+iv$ be the complex valued function of complex variable.

Analytic Function (Regular or Holomorphic Function)

A function $w = f(z)$ defined at z_0 is analytic at z_0 if it has a derivative at z_0 and at every point in some neighbourhood of z_0.

Necessary Condition for f(z) to be analytic

1) u_x ; v_x ; u_y ; v_y exist.

2) $u_x = v_y$; $v_x = -u_y$ (C-R equation).

Sufficient condition for f(z) to be analytic

1) u_x, v_x, u_y v_y are

① Is the function $f(z) = \bar{z}$ analytic ?

Sol

$f(z) = \bar{z}$

$\Rightarrow u+iv = \overline{x+iy} = x-iy$

$u = x$ $\qquad v = -y$

$u_x = 1$ $\qquad v_x = 0$

$u_y = 0$ $\qquad v_y = -1$

Here $u_x \neq v_y \Rightarrow$ CR equation is not satisfied.

$\Rightarrow f(z) = \bar{z}$ is analytic.

② Verify $f(z) = z^3$ is analytic or not.

Sol

$f(z) = z^3$

$u+iv = (x+iy)^3 = x^3 - iy^3 + 3x^2iy + 3xy^2$

$\Rightarrow u = x^3 - 3xy^2$ $\qquad v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ $\qquad v_x = 6xy$

③ Show that $|z|^2$ is not analytic at any point.

Sol

$f(z) = |z|^2$

$u+iv = |x+iy|^2 = x^2+y^2$

$u = x^2+y^2$ $\qquad v = 0$.

$u_x = 2x$ $\qquad u_y = 2y$

At $(0,0)$: $u_x = 0$, $u_y = 0$. $\qquad v_x = 0$, $v_y = 0$.

At $(0,0)$, $u_x = v_y$, $v_x = -u_y$ $\Rightarrow |z|^2$ is analytic at $(0,0)$.

From this, $|z|^2$ is not analytic at any point.

④ Prove that $w = \sin(2z)$ is an analytic function.

Proof

$u = \sin(2x)\cosh(2y)$ $\quad v = \cos(2x)\sinh(2y)$

$u_x = 2\cos(2x)\cosh(2y)$ $\quad v_x = -2\sin(2x)\sinh(2y)$

$u_y = 2\sin(2x)\sinh(2y)$ $\quad v_y = 2\cos(2x)\cosh(2y)$

$u_x = v_y$; $v_x = -u_y$

$\Rightarrow$ CR equ. is Satisfied.

$\Rightarrow$ $w = \sin(2z)$ is analytic.

(5) Test the analyticity of $f(z) = e^z$.

Sol

$f(z) = e^z$

$u+iv = e^{x+iy} = e^x[\cos y + i\sin y]$

$u = e^x \cos y$ $\quad v = e^x \sin y$

$u_x = e^x \cos y$ $\quad v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad v_y = e^x \cos y$.

(6) Test the analyticity of $f(z) = z^n$.

Sol

$$f(z) = z^n.$$

$$u+iv = (re^{i\theta})^n = r^n . e^{in\theta}$$

$$u+iv = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$u = r^n \cos(n\theta) \qquad v = r^n \sin(n\theta)$$

$$\frac{\partial u}{\partial r} = n.r^{n-1}\cos(n\theta) \qquad \frac{\partial v}{\partial r} = n.r^{n-1}\sin(n\theta)$$

$$\frac{\partial u}{\partial \theta} = -n.r^n \sin(n\theta) \qquad \frac{\partial v}{\partial \theta} = n.r^n.\cos(n\theta)$$

From this, $\frac{\partial u}{\partial r} = \frac{1}{r}.\frac{\partial v}{\partial \theta}$

$$\frac{\partial v}{\partial r} = \frac{-1}{r}.\frac{\partial v}{\partial \theta}$$

$\Rightarrow$ Cauchy – Riemann equation is satisfied

$\Rightarrow$ $f(z) = z^n$ is analytic.

(7) Find the constants a, b, c, if

$u = x + ay$ $\qquad v = bx + cy$

$u_x = 1$ $\qquad v_x = b$

$u_y = a$ $\qquad v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y \Rightarrow \boxed{1 = c}$

$v_x = -u_y \Rightarrow \boxed{b = -a}$

(8) Find the constants a, b if $f(z) = x + 2ay + i(3x + by)$ is analytic.

Sol

Given $f(z) = x + 2ay + i(3x + by)$

$u + iv = x + 2ay + i(3x + by)$

$u = x + 2ay$ $\qquad v = 3x + by$

$u_x = 1$ $\qquad v_x = 3$

$u_y = 2a$ $\qquad v_y = b.$