Critical points

Consider $w = f(z)$.

The critical points are the points at which

$$\frac{dw}{dz} = 0, \quad \text{and} \quad \frac{dz}{dw} = 0.$$

(1) Find the critical points of

$w^2 = (z-\alpha)(z-\beta)$.

Sol

$$w^2 = (z-\alpha)(z-\beta)$$

$$2w \cdot \frac{dw}{dz} = (z-\alpha) + (z-\beta) = 2z-\alpha-\beta$$

$$\frac{dw}{dz} = \frac{2z-\alpha-\beta}{2w} \quad (\&) \quad \frac{dz}{dw} = \frac{2w}{2z-\alpha-\beta}$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z-\alpha-\beta = 0 \Rightarrow z = \frac{\alpha+\beta}{2}$$

$$\frac{dz}{dw} = 0 \Rightarrow 2w = 0 \Rightarrow w = 0 \Rightarrow w^2 = 0$$

$$\Rightarrow (z-\alpha)(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

(2) Find the critical points of $w = z^2$.

Sol

$$w = z^2$$

$$\frac{dw}{dz} = 2z \qquad \frac{dz}{dw} = \frac{1}{2z}$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$$

$$\frac{dz}{dw} = 0 \Rightarrow \frac{1}{2z} = 0 \Rightarrow \boxed{z = \infty}$$

Conformal Mapping

A mapping $w = f(z)$ that preserves angle between ~~any~~ every pair of curves both in magnitude and direction is called conformal mapping.

~~Type (I) : ~~ ~~z+c~~

Sol

$$w = 1 + \frac{2}{z} = 1 + 2z^{-1}$$

$$\frac{dw}{dz} = -2z^{-2} = \frac{-2}{z^2} \qquad \frac{dz}{dw} = \frac{z^2}{-2}$$

The critical points are given by

$$\frac{dw}{dz} = 0 \Rightarrow \frac{-2}{z^2} = 0 \Rightarrow \boxed{z = \infty}$$

$$\frac{dz}{dw} = 0 \Rightarrow \frac{z^2}{-2} = 0 \Rightarrow \boxed{z = 0.}$$

Type (i) : $w = c + z$

(1) Find the image of $2x + y - 3 = 0$ under $w = z + 2i$.

Sol

$$w = z + 2i$$

$$u + iv = x + iy + 2i$$

$$u + iv = x + i(2 + y)$$

(9) Find the image of $|z| = 2$ under $w = z + 3 + 2i$.

Sol

$$w = z + 3 + 2i$$

$$u + iv = x + iy + 3 + 2i$$

$$u + iv = x + 3 + i(y + 2)$$

$u = x + 3$ $v = y + 2$

$x = u - 3$ $y = v - 2$

The image of $|z| = 2$ is

$|z| = 2 \Rightarrow |z|^2 = 4 \Rightarrow x^2 + y^2 = 4$

$\Rightarrow (u-3)^2 + (v-2)^2 = 4.$

Type (2) : $w = cz$

(1) Find the map of the circle $|z| = 3$ under the transformation $w = 2z$

Sol

$$w = 2z$$

The image of $|z| = 3$ is

$|z| = 3$

$\Rightarrow |z|^2 = 9$

$\Rightarrow x^2 + y^2 = 9$

$\Rightarrow \frac{u^2}{4} + \frac{v^2}{4} = 9 \Rightarrow u^2 + v^2 = 36.$

(2) Find the image of $0 \le x \le 2$ under $w = iz$.

Sol

$w = iz$

$\Rightarrow u + iv = i(x + iy)$

$\Rightarrow u + iv = ix - y$

$\Rightarrow u = -y \qquad v = x$

$\Rightarrow \boxed{y = -u} \qquad \boxed{x = v}$

Given $0 \le x \le 2$

$\Rightarrow 0 \le v \le 2$

Type (3) $W = e^z$

$\Rightarrow \log\rho + \log e^{i\varphi} = x+iy$

$\Rightarrow \log\rho + i\varphi = x+iy$

From this, $x = \log\rho \Rightarrow x = \frac{1}{2}\log(u^2+v^2)$

$y = \varphi \Rightarrow y = \tan^{-1}\left(\frac{v}{u}\right)$.

(1) Find the image of the straight line $y=x$ under $w=e^z$.

Sol

$w = e^z \Rightarrow x = \log\rho \;;\; y = \varphi$

$y = x \Rightarrow$ ~~$\varphi = \log\rho$~~

$\Rightarrow$ ~~$\tan^{-1}\left(\frac{v}{u}\right) =$~~ $\tan^{-1}\left(\frac{v}{u}\right) = \frac{1}{2}\log(u^2+v^2)$.

Type (4) – $w = z^2$

$w = z^2 \Rightarrow z = w^{1/2}$

$\Rightarrow x+iy = \left(\rho e^{i\varphi}\right)^{1/2} = \rho^{1/2}\cdot e^{i\varphi/2}$

$\Rightarrow x+iy = \rho^{1/2}\left[\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right]$

Cardioid $\rho = 2(1+\cos\varphi)$

Sol

$w = z^2 \Rightarrow x = \rho^{1/2}\cos\left(\frac{\varphi}{2}\right)$

$y = \rho^{1/2}\sin\left(\varphi/2\right)$.

$|z-1| = 1 \Rightarrow |x+iy-1| = 1$

$\Rightarrow |(x-1)+iy| = 1$

$\Rightarrow (x-1)^2+y^2 = 1$

$\Rightarrow x^2-2x+1+y^2 = 1$

$\Rightarrow x^2-2x+y^2 = 0$.

$\Rightarrow \rho - 2\rho^{1/2}\cos\left(\frac{\varphi}{2}\right) = 0$

$\Rightarrow \rho = 2\rho^{1/2}\cos\left(\frac{\varphi}{2}\right)$

$\Rightarrow \rho^2 = 4\rho\cos^2\left(\frac{\varphi}{2}\right)$

$\Rightarrow \rho = 4\cos^2\left(\frac{\varphi}{2}\right) = 2\left\{2\cos^2\left(\frac{\varphi}{2}\right)\right\}$

$\Rightarrow$

Type (5) – $w = \frac{1}{z}$

$$u+iv = \frac{1}{x+iy}$$

$$\Rightarrow x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

(1) Find the image of the half plane $x > c$, $c > 0$ under the transformation $w = \frac{1}{z}$. Show the regions graphically.

Sol

$$w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x+iy = \frac{1}{\quad} \times \frac{u-iv}{\quad}$$

$$x = \frac{u}{u^2+v^2} \quad ; \quad y = \frac{-v}{u^2+v^2}$$

$$x > c \Rightarrow \frac{u}{u^2+v^2} > c$$

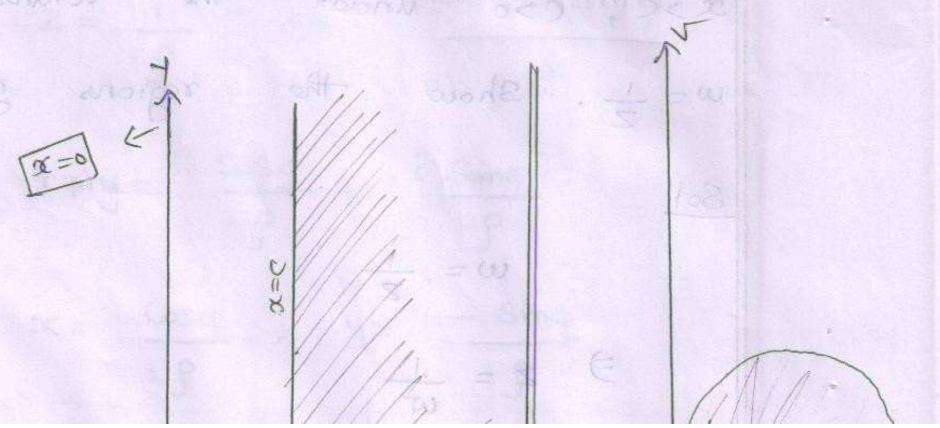
$$\Rightarrow \frac{u}{c} > u^2+v^2$$

$$\Rightarrow u^2+v^2-\frac{u}{c} < 0.$$

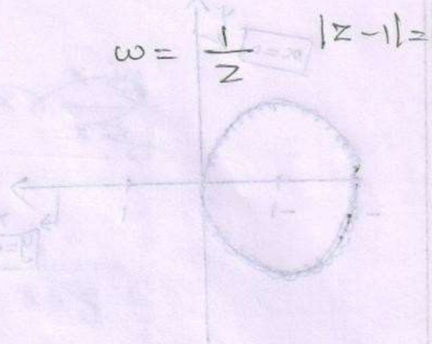
$$\Rightarrow u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2 + v^2 < \left(\frac{1}{2c}\right)^2$$

$$\Rightarrow \left(u-\frac{1}{2c}\right)^2 + v^2 < \left(\frac{1}{2c}\right)^2 \rightarrow \text{Inside of}$$

the circle whose center is $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2c}$.



(8) Find the image of $|z+1|=1$ under the transformation $w = \frac{1}{z}$



Sol

$$w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

$$|z+1| = 1 \Rightarrow |x+iy+1| = 1$$

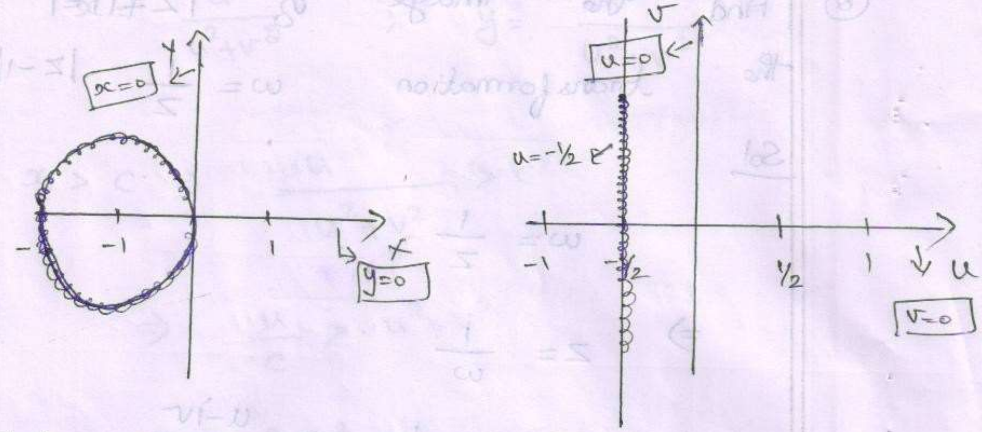
$$\Rightarrow |(x+1)+iy| = 1$$

$$\Rightarrow (x+1)^2 + y^2 = 1 \rightarrow$$ Circle with center $(-1,0)$ & radius 1.

$$\Rightarrow x^2 + 2x + 1 + y^2 = 1$$

$$\Rightarrow x^2 + 2x + y^2 = 0.$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{2u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 0.$$



③ Find the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$

Sol

$$w=\frac{1}{z}$$

$$\Rightarrow \quad z=\frac{1}{w}=\frac{1}{\rho e^{i\varphi}}$$

$$\Rightarrow \quad z=\frac{1}{\rho}e^{-i\varphi}=\frac{\cos\varphi-i\sin\varphi}{\rho}$$

$$\Rightarrow \quad x+iy=\frac{\cos\varphi}{\rho}-i\frac{\sin\varphi}{\rho}$$

$$\Rightarrow \quad x=\frac{\cos\varphi}{\rho}\ ; \quad y=\frac{-\sin\varphi}{\rho}$$

(4) Find the image in the W-plane of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $\omega = \frac{1}{z}$.

Sol

$$\omega = \frac{1}{z}$$

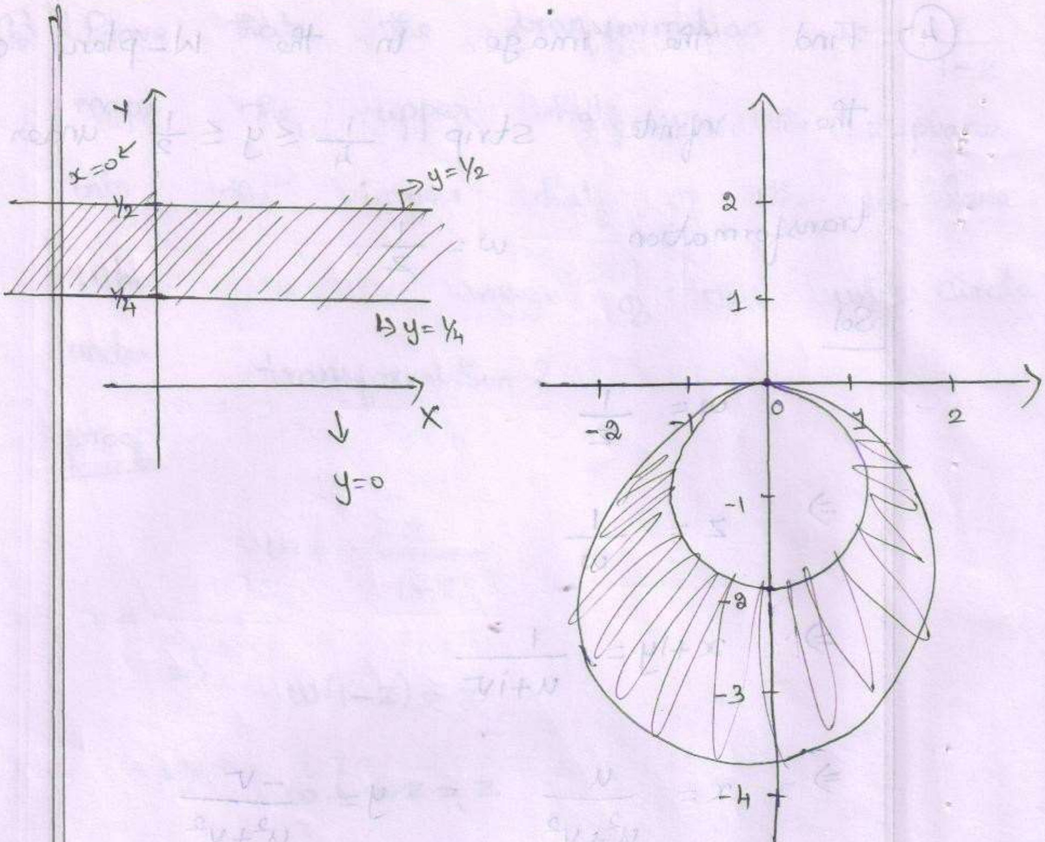
$$\Rightarrow z = \frac{1}{\omega}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \; ; \quad y = \frac{-v}{u^2+v^2}$$

$$\frac{1}{4} \le y \le \frac{1}{2}$$

$\frac{1}{4} \le y$	$y \le \frac{1}{2}$
$\frac{1}{4} \le \frac{-v}{u^2+v^2}$	$\frac{-v}{u^2+v^2} \le \frac{1}{2}$
$\Rightarrow u^2+v^2 \le -4v$	$u^2+v^2 \ge -2v$



(5) Show that the transformation $w = \frac{1}{z}$ transforms all circles and straight lines in the z-plane into circles or straight lines in the w-plane.

Sol

$$w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv}$$

(1) represents a circle when $a \neq 0$.

(1) represents a st. line when $a = 0$.

$$(1) \Rightarrow a\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0.$$

$$\Rightarrow \frac{a}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0.$$

$$\Rightarrow c(u^2+v^2) + 2gu - 2fv + a = 0 \quad \text{--- (2)}$$

(2) represents a circle when $c \neq 0$.

(2) represents a st. line when $c = 0$.

I : $a \neq 0$; $c \neq 0$.

Circle is mapped onto a circle.

II $a \neq 0$, $c = 0$

Circle is mapped onto st. line

III $a = 0$, $c \neq 0$.

(6) Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane. What is the image of the unit circle under transformation?

Proof

$$w = \frac{z}{1-z}$$

$$\Rightarrow \quad w(1-z) = z$$

$$\Rightarrow \quad w - wz = z$$

$$\Rightarrow \quad z + wz = w$$

$$\Rightarrow \quad z(1+w) = w$$

$$\Rightarrow \quad z = \frac{w}{1+w}$$

$$\Rightarrow \quad x+iy = \frac{[u+iv]}{[(1+u)+iv]} \times \frac{[(1+u)-iv]}{[(1+u)-iv]}$$

$$= u(1+u)+v^2 + i[(1+u)v - uv]$$

✱ Upper half of z-plane : $y > 0$.

$$\Rightarrow \frac{v}{(1+u)^2 + v^2} > 0$$

$$\Rightarrow v > 0$$

↳ Upper half of W-plane.

✱✱ Unit circle : $|z| = 1$

$$\left|\frac{\omega}{1+\omega}\right| = 1$$

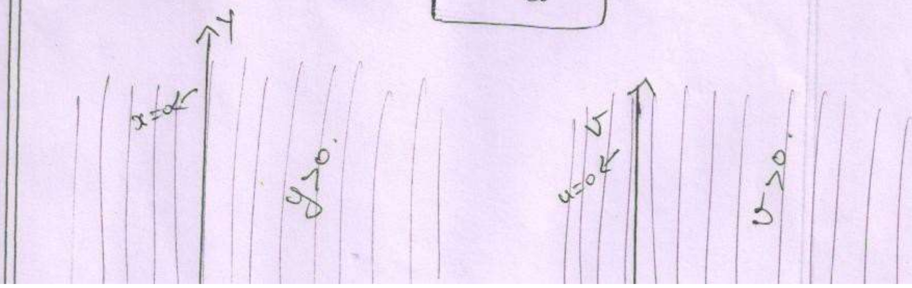
$$\Rightarrow |\omega| = |1+\omega|$$

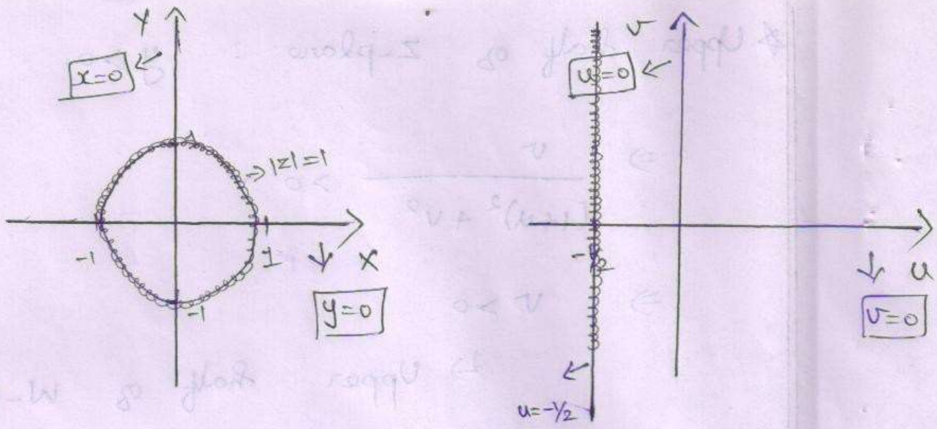
$$\Rightarrow |u+iv| = |(1+u)+iv|$$

$$\Rightarrow u^2+v^2 = (1+u)^2+v^2$$

$$\Rightarrow u^2+v^2 = 1+u^2+2u+v^2$$

$$\Rightarrow 2u+1=0 \Rightarrow \boxed{u = \frac{-1}{2}}$$




y
x=0
|z|=1
-1
1
-1
x
y=0
v
u=0
-1/2
u
v=0
u=-1/2