### Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\ \vec{F} = 0$ (i.e)$\nabla . \vec{F} = 0$

### Curl of a vector function

If $\vec{F}(x, y, z)$ is a differentiable vector point function defines at each point $(x, y, z)$ in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

**Note:** $\nabla \times \vec{F}$ Is a vector point function.

### Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.e)$ $\nabla \times \vec{F} = 0$

### Scalar potential

If $\vec{F}$ is an irrotational vector, then there exists a scalar function $\varphi$ such that $\vec{F} = \nabla\varphi$. Such a scalar function is called scalar potential of $\vec{F}$.

## Problems based on Divergence and Curl of a vector

**Example: 2.21 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then find div $\vec{r}$ and curl$\vec{r}$**

**Solution:**

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now div $\vec{r} = \nabla \cdot \vec{r}$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

And curl $\vec{r} = \nabla \times \vec{r}$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}.$$

**Example: 2.22 If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla . \vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).**

**Solution:**

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla . \vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla . \vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3 + 2) + \vec{k}(-4 + 2)$$

$$= -\vec{i} - 2\vec{k}$$

**Example: 2.23 If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$ , $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).**

**Solution:**

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x+y)\vec{i} - (2x - 2x)\vec{j} + (y+z)\vec{k}$$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{i} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{i} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{i} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x+y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y+z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x+y) & 0 & y+z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{i} + \vec{k}$

**Example: 2.24 Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = grad(x^3+y^3+z^3-3xyz)$**

**Solution:**

Given $\vec{F} = grad(x^3+y^3+z^3-3xyz)$

$$= \vec{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\vec{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)+\vec{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = \vec{i}(3x^2-3yz)+\vec{j}(3y^2-3xz)+\vec{k}(3z^2-3xy)$$

Now $div\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$

$$= 6x+6y+6z$$

$$= 6(x+y+z)$$

$$Curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \vec{i}[-3x+3x]-\vec{j}[-3y+3y]+\vec{k}[-3z+3z]$$

$$= \vec{0}$$

**Example: 2.25 Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$**

**Solution:**

Given $\varphi = x^2y^3z^4$

$$grad\,\varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4)+\vec{j}(x^2 3y^2z^4)+\vec{k}(x^2y^3 4z^3)$$

$$Div(grad\,\varphi) = \nabla\cdot(grad\,\varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4)+\frac{\partial}{\partial y}(x^2 3y^2z^4)+\frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4+6x^2yz^4+12x^2y^3z^4$$

$$\therefore Div(grad\,\varphi)_{(1,1,1)} = 2+6+12 = 20$$

$$Curl(grad\,\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3-12x^2y^2z^3)-\vec{j}(8xy^3z^3-8xy^3z^3)+\vec{k}(6xy^2z^4-6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore Curl\,grad\varphi_{(1,1,1)} = \vec{0}$$

**Vector Identities**

1) $\nabla\cdot(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$
2) $\nabla\times(\varphi\,\vec{F}) = \varphi\left(\nabla\times\vec{F}\right) + (\nabla\varphi)\times\vec{F}$
3) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$
4) $\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$
5) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) - (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) - (\vec{B}\cdot\nabla)\vec{A}$
6) $\nabla\cdot(\nabla\varphi) = \vec{0}$
7) $\nabla\cdot(\nabla\times\vec{F}) = 0$
8) $\nabla\times(\nabla\times\vec{F}) = \nabla\left(\nabla\cdot\vec{F}\right) - \nabla^2\vec{F}$
9) $\nabla\cdot\nabla\varphi = (\nabla\cdot\nabla)\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a laplacian operator

**1) If $\varphi$ is a scalar point function, $\vec{F}$ is a vector point function, then $\nabla.(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$**

**Proof:**

$$\nabla\cdot(\varphi\,\vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\varphi\,\vec{F})$$

$$= \sum\vec{i}\cdot\frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$= \sum\vec{i}\cdot\left(\varphi\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right)$$

$$= \varphi\left(\sum\vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right) + \vec{F}\cdot\left(\sum\vec{i}\frac{\partial\varphi}{\partial x}\right)$$

$$\therefore\nabla\cdot(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$$

**2) If $\varphi$ is a scalar point fuction, $\vec{F}$ is a vector point function, then$\nabla\times(\varphi\,\vec{F}) = \varphi\left(\nabla\times\vec{F}\right) + (\nabla\varphi)\times\vec{F}$**

**Proof:**

$$\nabla\times(\varphi\,\vec{F}) = \sum\vec{i}\times\frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$=\sum\vec{i}\times\left[\varphi\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right]$$

$$=\sum\vec{i}\times\left(\frac{\partial\varphi}{\partial x}\vec{F} + \varphi\frac{\partial\vec{F}}{\partial x}\right)$$

$$=\left(\sum\vec{i}\frac{\partial\varphi}{\partial x}\right)\times\vec{F} + \varphi\left[\sum\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right]$$

$$\therefore\nabla\times(\varphi\,\vec{F}) = \nabla\varphi\times\vec{F} + \varphi(\nabla\times\vec{F})$$

**3) If $\vec{A}$ and $\vec{B}$ are vector point functions, then $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$**

**Proof:**

$$\nabla . (\vec{A} \times \vec{B}) = \sum \vec{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= -\left(\sum \vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) \cdot \vec{A} + \left(\sum \vec{i} \times \frac{\partial \vec{A}}{\partial x}\right) . \vec{B}$$

$$= -(\nabla \times \vec{B}) . \vec{A} + (\nabla \times \vec{A}) \cdot \vec{B}$$

$$\therefore \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \qquad \left[\because (\nabla \times \vec{A}) \cdot \vec{B} = \vec{B} \cdot (\nabla \times \vec{A})\right]$$

**(4) If $\vec{A}$ and $\vec{B}$ are vector point functions, then**

**$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$**

**Proof:**

$$\nabla \times (\vec{A} \times \vec{B}) = \sum \vec{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \sum \vec{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

$$\nabla \times (\vec{A} \times \vec{B}) = \sum \left[(\vec{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum \left[\left(\vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\vec{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right]$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \sum(\vec{B} \cdot \vec{i})\frac{\partial \vec{A}}{\partial x} - \sum(\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\vec{B} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{A} - \left(\vec{A} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\therefore \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

**(6) If $\varphi$ is a scalar point function, then $\nabla \times (\nabla\varphi) = \vec{0}$.**

**(or)**

**Prove that $curl(grad\ \varphi) = 0$.**

**Solution:**

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla \times \nabla\varphi = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{\imath}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right]$$

$$= \sum \vec{\imath}\,(\vec{0}) = \vec{0}$$

**Prove that $div\left(curl\,\vec{F}\right) = 0$.**

**Solution:**

Let $\vec{F} = F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot$$

$$\left[\vec{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0$$

**(9) $\nabla \cdot (\nabla\varphi) = (\nabla \cdot \nabla)\,\varphi = \nabla^2\varphi$**

**Proof:**

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore\ \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

**Example: 2.26 Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$**

**Solution:**

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}$$

**Example: 2.28 If is a constant vector and is the position vector of any point, prove that**

**(i) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$ (ii) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$**

**Solution:**

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$

$= 0 + 0 + 0 = 0$

$$\text{(ii) } \nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_3 x \end{vmatrix}$$

`$= \vec{i}(a_1 + a_1) - \vec{j}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$

$= 2a_1\vec{i} + 2a_2\vec{j} + 2a_3\vec{k}$

$= 2(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 2\vec{a}$

**Example: 2.29 Prove that $\boldsymbol{curl(f(r)\vec{r}) = \vec{0}}$**

**Solution:**

Let $f(r)\vec{r} = f(r)[x\,\vec{i} + y\,\vec{j} + z\,\vec{k}]$

$= xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$= \sum \vec{i}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$

$= \sum \vec{i}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$

$= \sum \vec{i}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$

$= \sum \vec{i}\,(0)$

$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$

**Example: 2.30 Prove that $curl[\varphi\,\nabla\varphi] = \vec{0}$**

**(or)**

**Prove that $\nabla \times [\varphi\,\nabla\varphi] = \vec{0}$**

**Solution:**

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum\vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum\vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum\vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$