2.4 Green's Theorem

Green's theorem relates a line integral to the double integral taken over the region bounded by the closed curve.

Statement

If $M(x,y)$ and $N(x,y)$ are continuous functions with continuous, partial derivatives in a region R of the xy – plane bounded by a simple closed curve C, then

$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$, where C is the curve described in the positive direction.

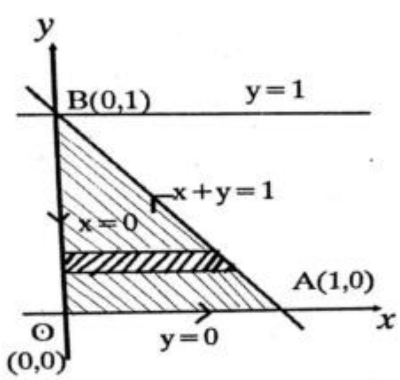
Vector form of Green's theorem

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dR$$

Problems based on Green's theorem

Example: 2.64 Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

Solution:



We have to prove that $\int_{c} M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_{c} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_{c} M\,dx + N\,dy$$

By Green's theorem in the plane,

$$\int_{c} M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,dx$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3} \ldots (1)$$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

Along $OA, y = 0 \Rightarrow dy = 0, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along $AB, y = 1 - x \Rightarrow dy = -dx$ *and* x varies from 1 to 0

$$\therefore \int_{AB} M\,dx + N\,dy = \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]dx$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0$ *and* y varies from 1 to 0

$$\therefore \int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\therefore \int_c M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \dots(2)$$

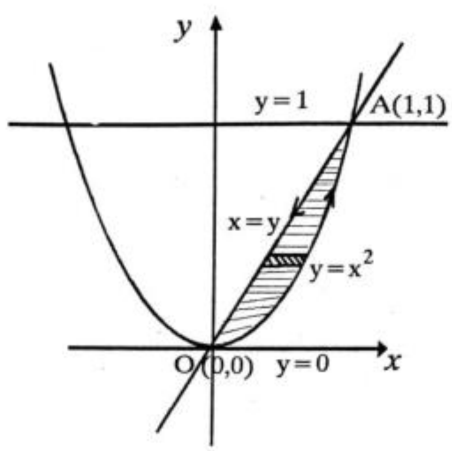
$\therefore$ From (1) and (2)

$$\therefore \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Example: 2.65 Verify Green's theorem in the XY –plane for $\int_c (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = xy + y^2$ *and* $N = x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Limits:

x varies from y to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_y^{\sqrt{y}} 2x - (x + 2y)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y\sqrt{y}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{\frac{3}{2}} + 3\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

L.H.S = $\int_c M\, dx + N\, dy$

Consider $\int M\, dx + N\, dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\, dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\, dx + N\, dy = \int_0^1 [(x(x^2) + (x^2)^2) dx + x^2 \cdot 2x\, dx]$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along $AO, y = x \Rightarrow dy = dx$ and x varies from 1 to 0

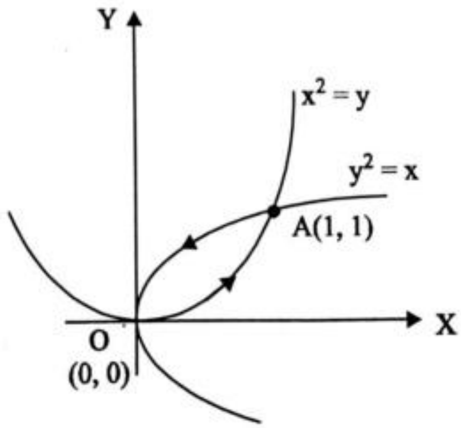
$$\therefore \int_{AO} M\, dx + N\, dy = \int_1^0 (x^2 + x^2) dx + x^2\, dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1$$

L.H.S = $\int M\, dx + N\, dy = \frac{19}{20} - 1 = -\frac{1}{20}$

Example: 2.66 Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $y = x^2, x = y^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y^2 to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\, dx\, dy$$

$$= \int_0^1 [10xy]_{y^2}^{\sqrt{y}} dy$$

$$= 10 \int_0^1 (y\sqrt{y} - y^3)\, dy$$

$$= 10 \left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{3}{2}$$

L.H.S $= \int_c M\, dx + N\, dy$

Consider $\int M\, dx + N\, dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\, dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\, dx + N\, dy = \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)(2x) dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) dx$$

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2)\, dx$$

$$= \left[-20\frac{x^5}{5} + 8\,\frac{x^4}{4} + 3\frac{x^3}{3}\right]_0^1$$

$$= -4 + 2 + 1 = -1$$

Along $AO, x = y^2 \Rightarrow dx = 2ydy \; and \; y$ varies from 1 to 0

$$\therefore \int_{Ao} M\,dx + N\,dy = \int_1^0 (3y^4 - 8y^2)2ydy + (4y - 6yy^2)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)dx$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dx$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_1^0$$

$$= 0 - \left[1 - \frac{11}{2} + 2\right]$$

$$= -\left(3 - \frac{11}{2}\right) = \frac{5}{2}$$

L.H.S = $\int_c M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.67 Verify Green's theorem in the plane for the integral $\int_c (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = x - 2y \; and \; N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -2 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 + 2)dxdy$$

$$= 3\iint_R dxdy$$

$= 3$ (Area of the circle)

$$= 3\pi r^2$$

$$= 3\pi \quad (\because radius = 1)$$

L.H.S = $\int_c M\,dx + N\,dy$

Given C is $x^2 + y^2 = 1$

The parametric equation of circle is

$x = \cos\theta,\; y = \sin\theta$

$dx = -\sin\theta d\theta,\; dy = \cos\theta\, d\theta$

Where θ varies from 0 to 2π

$$\therefore \int_c M\,dx + N\,dy = \int_0^{2\pi} (\cos\theta - 2\sin\theta)\,(-\sin\theta\, d\theta) + \cos\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + \sin^2\theta + 1)\, d\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \int_0^{2\pi} \left(-\frac{\sin 2\theta}{2} + \frac{1-\cos 2\theta}{2} + 1\right) d\theta$$

$$= \left[-\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right) + \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + \theta\right]_0^{2\pi}$$

$$= \left[\frac{\cos(4\pi)}{4} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} + 2\pi\right] - \left[\frac{\cos 0}{4} + \frac{0}{2} - \frac{\sin 0}{4} + 0\right]$$

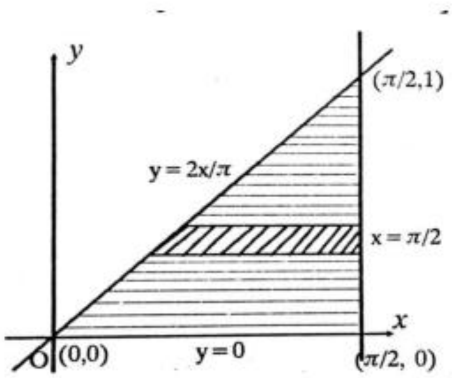
$$= \frac{1}{4} + \pi + 2\pi - \frac{1}{4} = 3\pi \quad [\because \sin n\pi = 0, \sin 0 = 0, \cos 0 = 1], [\cos n\pi = (-1)^n]$$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.68 Using Green's theorem evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$.

Solution:



We have to prove that $\int_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Here, $M = y - \sin x$ and $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0 \qquad \Rightarrow \frac{\partial N}{\partial x} = -\sin x$

Limits:

x varies from $\frac{y\pi}{2}$ to $\frac{\pi}{2}$

y varies from 0 to 1

Hence $\int_c (y - \sin x)dx + \cos x\, dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\ dx\, dy$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\ dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2}\right)\right] dy$$

$$= \int_0^1 \left[0 - \frac{\pi}{2} - \cos\frac{y\pi}{2} + \frac{y\pi}{2}\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}\frac{y^2}{2}\right]_0^1$$

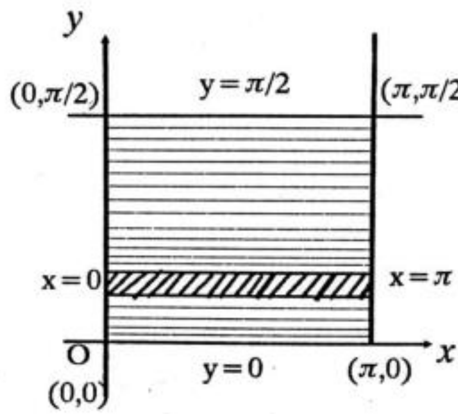
$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Example: 2.69 Evaluate by Green's theorem $\int_c [e^{-x}(\sin y\, dx + \cos y\, dy)]$ where C being the rectangle with vertices $(0,0), (\pi,0), \left(\pi,\frac{\pi}{2}\right)$ and $\left(0,\frac{\pi}{2}\right)$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = e^{-x}\sin y$ and $N = e^{-x}\cos y$

$\Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y \qquad \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

Limits:

x varies from 0 to π

y varies from 0 to $\frac{\pi}{2}$

$\therefore \int_c [e^{-x}(\sin y\, dx + \cos y\, dy)] = \int_0^{\frac{\pi}{2}}\int_0^{\pi}(-e^{-x}\cos y - e^{-x}\cos y)\, dxdy$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\pi} -2\,e^{-x}\cos y\;dx\,dy$$

$$= -2\int_0^{\frac{\pi}{2}}\left[\frac{e^{-x}\cos y}{-1}\right]_0^{\pi} dy$$

$$= 2\int_0^{\frac{\pi}{2}}[e^{-\pi}\cos y - e^{0}\cos y]dy$$

$$= 2\int_0^{\frac{\pi}{2}}[e^{-\pi}\cos y - \cos y]dy$$

$$= 2\,[e^{-\pi}\sin y - \sin y]_0^{\frac{\pi}{2}}$$

$$= 2\left[\left(e^{-\pi}\sin\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (e^{-\pi}\sin 0 - \sin 0)\right]$$

$$= 2\,[e^{-\pi} - 1]$$

Example: 2.70 Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_c (xdy - ydx)$. Hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

Solution:

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = -y \; and \; N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$

$\therefore \int_c (xdy - ydx) = \iint_R (1+1)\,dx\,dy$

$= 2\iint_R dx\,dy = 2$ (Area enclosed by C)

$\therefore$ Area enclosed by $C = \frac{1}{2}\int_c (xdy - ydx)$

Equation of ellipse in parametric form is $x = a\cos\theta \; and \; y = b\sin\theta$ where $0 \le \theta \le 2\pi$.

$\therefore$ Area of the ellipse $= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\; d\theta$

$= \frac{1}{2}\, ab \int_0^{2\pi} (cos^2\theta + sin^2\theta)\, d\theta$

$= \frac{1}{2}\, ab \int_0^{2\pi} d\theta = \frac{1}{2}\, ab\, [\theta]_0^{2\pi} = \pi ab$

Exercise: 2.4

1. Using Green's theorem in the plane, evaluate $\int_c (x^2 - y^2)dx + 2xydy$ where C is the closed curve of the region bounded by $y = x^2 and\; y^2 = x$ **Ans:** $\frac{3}{5}$

2. Find by Green's theorem the value of $\int_c (x^2ydx + ydy)$ along the closed curve formed by $y = x^2 and\; y^2 = x$ between $(0,0)\; to\; (1,1)$ **Ans:** $\frac{1}{28}$

3. Verify Green's theorem for the integral $\int_c [(x-y)dx+(x+y)dy]$ taken around the boundary area in the first quadrant between the curves $y = x^2 and\ y^2 = x$.

Ans: Common value $= \frac{2}{3}$

4. Find the area of a circle of radius 'a' using Green's theorem. **Ans:** πa^2

5. Evaluate $\int_c [(sinx - y)dx - \cos x\, dy]$, where C is the triangle with vertices $(0,0), \left(\frac{\pi}{2}, 0\right)\ and\ \left(\frac{\pi}{2}, 1\right)$ **Ans:** $\frac{2}{\pi} + \frac{\pi}{4}$

6. Using Green's theorem, find the value of $\int_c [(xy - x^2)dx + x^2 y dy]$ along the closed curve C formed by $y = 0, x = 1\ and\ y = x$ **Ans:** $-\frac{1}{12}$

7. Verify Green's theorem for $\int_c [(x^2 - y^2)dx + 2xydy]$, where C is the boundary of the rectangle in the xoy – plane bounded by the lines $x = 0, x = a, y = 0\ and\ y = b$.

Ans: Common value $= 2ab^2$

8. Verify Green's theorem for $\int_c [(2x - y)dx + (x + y)dy]$, where C is the boundary of the Circle $x^2 + y^2 = a^2$ in the xoy – plane. **Ans:** $2\pi a^2$