# Type:II Problem on Triple Integral if region is given

**Example: 4.51**

**Express the region $x \geq 0, y \geq 0, z \geq o, x^2 + y^2 + z^2 \leq 1$ by triple integration.**

**Solution:**

For the given region, z varies from 0 to $\sqrt{1 - x^2 - y^2}$

y varies from 0 to $\sqrt{1 - x^2}$

x varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

**Example: 4.52**

**Evaluate $\iiint x^2yzdxdydz$ taken over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.**

**Solution:**

Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Limits are , z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$\iiint x^2yzdxdydz = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2 dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2y\frac{z^2}{2}\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(\frac{x^2yc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a \int_0^{bk} x^2y\left(k - \frac{y}{b}\right)^2 dydx \qquad \left[\because k = 1 - \frac{x}{a}\right]$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bk} x^2y\left(k^2+\frac{y^2}{b^2}-\frac{2ky}{b}\right)dydx$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bk} x^2\left(yk^2+\frac{y^3}{b^2}-\frac{2ky^2}{b}\right)dydx$$

$$= \frac{c^2}{2}\int_0^a x^2\left[\frac{k^2y^2}{2}+\frac{y^4}{4b^2}-\frac{2ky^3}{3b}\right]_0^{bk}dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2}+\frac{b^4k^4}{4b^2}-\frac{2b^3k^4}{3b}\right)dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2}+\frac{b^2k^4}{4}-\frac{2b^2k^4}{3}\right)dx$$

$$= \frac{b^2c^2}{2}\int_0^a k^4x^2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)dx$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1-\frac{x}{a}\right)^4dx$$

$$\left[\because (1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\right]$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1-\frac{4x}{a}+\frac{4\times3}{2!}\times\frac{x^2}{a^2}-\frac{4\times3\times2}{3!}\times\frac{x^3}{a^3}+\frac{4\times3\times2\times\times1}{4!}\times\frac{x^4}{a^4}\right)dx$$

$$= \frac{b^2c^2}{24}\int_0^a\left(x^2-\frac{4x^3}{a}+\frac{6x^4}{a^2}-\frac{4x^5}{a^3}+\frac{x^6}{a^4}\right)dx$$

$$= \frac{b^2c^2}{24}\left[\frac{x^3}{3}-\frac{4x^4}{4a}+\frac{6x^5}{5a^2}-\frac{4x^6}{6a^3}+\frac{x^7}{7a^4}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3}-\frac{a^4}{a}+\frac{6a^5}{5a^2}-\frac{2a^6}{3a^3}+\frac{a^7}{7a^4}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3}-a^3+\frac{6a^3}{5}-\frac{2a^3}{3}+\frac{a^3}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[\frac{1}{3}-1+\frac{6}{5}-\frac{2}{3}+\frac{1}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{35-105+126-70+15}{105}\right)$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{1}{105}\right)$$

$$= \frac{a^3b^2c^2}{2520}$$

**Example: 4.53**

**Find the value of $\iiint xyz\,dxdydz$ through the positive spherical octant for which $x^2+y^2+z^2 \le a^2$**

**Solution:**

In the positive octant, the limits are

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

$$I= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyzdzdydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2)\, dydx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dydx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2 - x^2)\left[a^2 - x^2 - \frac{(a^2-x^2)}{2}\right] dx$$

$$= \frac{1}{2}\int_0^a \frac{x(a^2-x^2)(a^2-x^2)}{4} dx$$

$$= \frac{1}{8}\int_0^a x(a^2 - x^2)^2\, dx$$

Put $a^2 - x^2 = t$ $\qquad$ $x = 0 \rightarrow t = a^2$

$-2xdx = dt$ $\qquad$ $x = a \rightarrow t = 0$

$$\Rightarrow \quad I = \frac{1}{8}\int_{a^2}^0 t^2 \left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{16}\int_{a^2}^0 t^2 dt$$

$$= \frac{1}{16}\int_0^{a^2} t^2 dt$$

$$= \frac{1}{16}\left[\frac{t^3}{3}\right]_0^{a^2}$$

$$= \frac{1}{16}\left(\frac{a^6}{3}\right)$$

$$= \frac{a^6}{48}$$

**Example: 4.54**

**Evaluate $\iiint_D (x + y + z)\, dxdydz$ where $D: 1 \le x \le 2, 2 \le y \le 3, 1 \le z \le 3$**

**Solution:**

$$\iiint_D (x + y + z)\, dxdydz = \int_1^2 \int_2^3 \int_1^3 (x + y + z)\, dzdydx$$

$$= \int_1^2 \int_2^3 \left[xz + yz + \frac{z^2}{2}\right]_1^3 dydx$$

$$= \int_1^2 \int_2^3 (3x + 3y + \frac{9}{2} - x - y - \frac{1}{2})\, dydx$$

$$= \int_1^2 \int_2^3 (2x + 2y + 4)\, dydx$$

$$= \int_1^2 \left[2xy + \frac{2y^2}{2} + 4y\right]_2^3 dx$$

$$= \int_1^2 (6x + 9 + 12 - 4x - 4 - 8)\, dx$$

$$= \int_1^2 (2x + 9)\, dx$$

$$= \left[\frac{2x^2}{2} + 9x\right]_1^2$$

$$= 4 + 18 - 1 - 9$$

$$= 12$$

**Example: 4.55**

**Evaluate $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2$**

**Solution:**

$$\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{\sqrt{(a^2-x^2-y^2)^2}-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (\sin^{-1}1 - 0)\, dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2} dydx$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi}{2}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - 0\right]$$

$$= \frac{\pi}{2}\frac{a^2}{2}\frac{\pi}{2}$$

$$= \frac{\pi^2 a^2}{8}$$