4.5 Area enclosed by plane curves (Cartesian coordinates)

Area = $\int\int dy\,dx$ (or) Area = $\int\int dxdy$

Example: 4.28

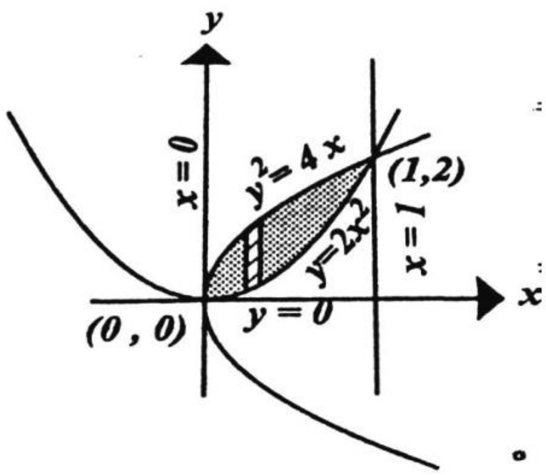
Find the area enclosed by the curves y=$2x^2$ and $y^2 = 4x$

Solution:

Area = $\int\int dy\,dx$

$y:\ 2x^2 \to 2\sqrt{x}$

$x:\ 0 \to 1$



$$\text{Area} = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\,dx$$

$$= \int_0^1 [y]_{2x^2}^{2\sqrt{x}} dx$$

$$= \int_0^1 (2\sqrt{x} - 2x^2)\,dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$$

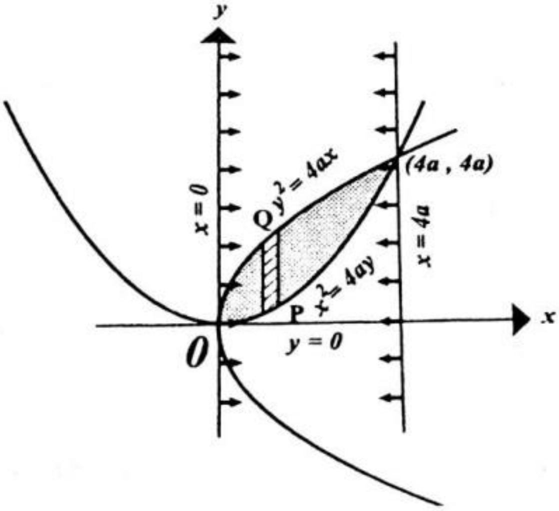
$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

Example: 4.29

Find the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$

Solution:



Area $= \int \int dy\, dx$

$$y : \frac{x^2}{4a} \rightarrow 2\sqrt{ax}$$

$$x : 0 \rightarrow 4a$$

$$= \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$$

$$= \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} (2\sqrt{ax} - \frac{x^2}{4a})\, dx$$

$$= \left[\frac{2\sqrt{a}\; x^{3/2}}{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\, (4a)^{3/2} - \frac{(4a)^3}{12a}$$

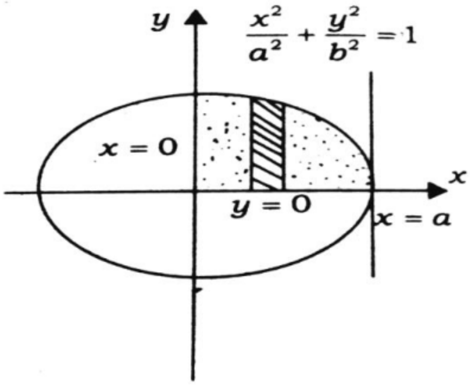
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Example: 4.30

Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:



Area $= 4 \iint dx\, dy$

$x : 0 \rightarrow \frac{a}{b}\sqrt{b^2 - y^2}$

$y : 0 \rightarrow ab$

$$\text{Area} = 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy\, dx$$

$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$

$= 4\int_0^b \left[\frac{a}{b}\sqrt{b^2 - y^2} - 0\right] dy$

$= \frac{4a}{b}\left[\frac{b^2}{2} sin^{-1}\left(\frac{y}{b}\right) + \frac{y}{2}\sqrt{b^2 - y^2}\right]_0^b$

$= \frac{4a}{b}[\left(\frac{b^2}{2}\frac{\pi}{2} + 0\right) - 0]$

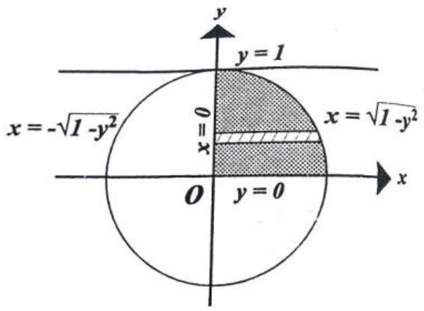
$= \frac{4ab}{b}\frac{b^2}{2}\frac{\pi}{2}$

$= \pi ab$

Example: 4.31

Evaluate $\iint xy\ dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$

Solution:



$x : 0 \rightarrow \sqrt{1 - y^2}$

$y : 0 \rightarrow 1$

$$\iint xy\ dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{2}\int_0^1 (\sqrt{1 - y^2})^2 y\,dy$$

$$= \frac{1}{2}\int_0^1 (1 - y^2)\,y\,dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\,dy$$

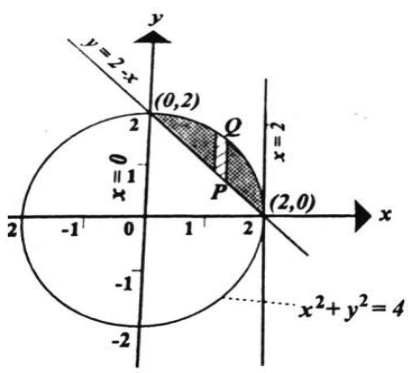
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{2}\left[\frac{1}{4}\right] = \frac{1}{8}$$

Example: 4.32

Find the smaller of the area bounded by $y = 2 - x$ and $x^2 + y^2 = 4$

Solution:



Area= $\iint dy\,dx$

$y:\ 2-x \rightarrow \sqrt{4-x^2}$

$x:\ 0 \rightarrow 2$

$$= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 [\sqrt{4-x^2} - (2-x)]dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + \frac{4}{2}\left(\frac{\pi}{2}\right) - 4 + \frac{4^2}{2}$$

$$= \pi - 2 \text{ square unit}$$

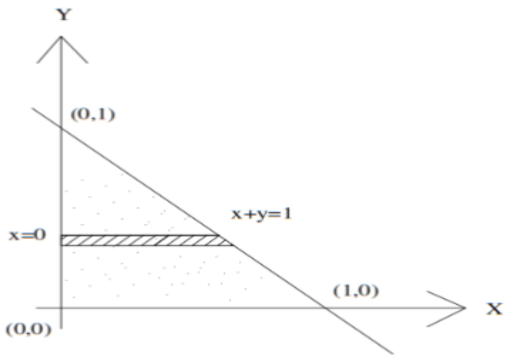
Example: 4.33

Evaluate $\iint xy\,dxdy$ over the positive quadrant for which $x + y \le 1$

Solution:

$x:\ 0 \rightarrow 1-y$

$y:\ 0 \rightarrow 1$



$$\iint xy\,dxdy = \int_0^1 \int_0^{1-y} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2 y}{2}\right)_0^{1-y} dy$$

$$= \int_0^1 \frac{1-y^2 y}{2} dy$$

$$= \frac{1}{2}\int_0^1 (y^2 - 2y^2 + y^3)dy$$

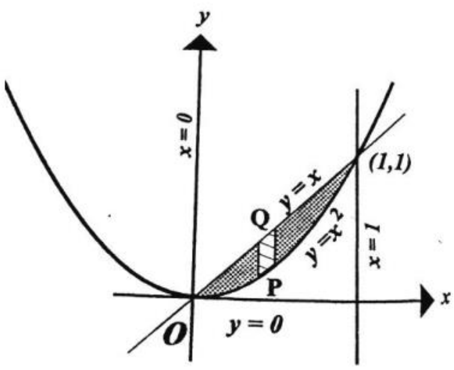
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] \quad = \frac{1}{2}\left[\frac{16-8+3}{12}\right] \quad = \frac{1}{24}$$

Example: 4.34

Using double integral find the area bounded by y = x and y = x^2

Solution:



$$\text{Area} = \iint dy\,dx$$

$$y : x^2 \to x$$

$$x : 0 \to 1$$

$$= \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 [y]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x - x^2)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

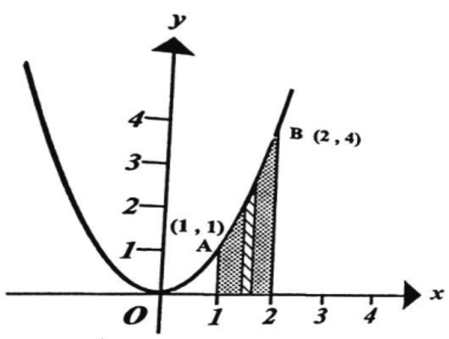
Example: 5.35

Evaluate $\iint (x^2 + y^2)dxdy$ where A is area bounded by the curves x^2=y, x=1 and x=2 about x axis

Solution:

$$y : 0 \to x^2$$

$$x : 1 \to 2$$



$$\iint (x^2 + y^2)dxdy = \int_1^2 \int_0^{x^2} (x^2 + y^2)dydx$$

$$= \int_1^2 \left[x^2y + \frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_1^2 (x^4 + \frac{x^6}{3})dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{21}\right]_1^2$$

$$= [\frac{2^5}{5} + \frac{2^7}{21} - \frac{1}{5} - \frac{1}{21}]$$

$$= \frac{1286}{105}$$

Example: 4.36

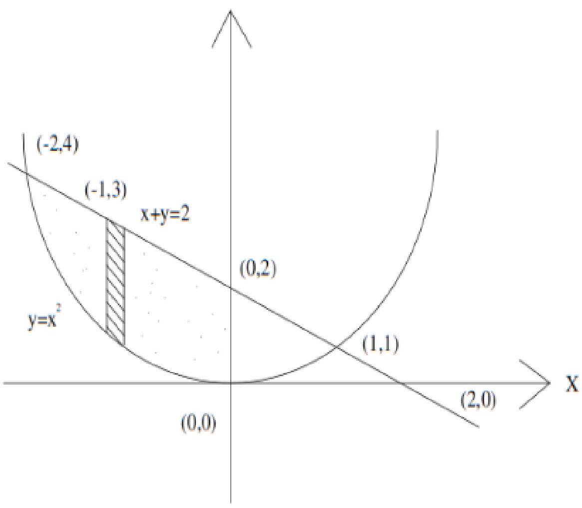
Find the area enclosed by the curves $y = x^2$ and $x + y - 2 = 0$

Solution:

Given y=x^2 and $x + y - 2 = 0$

x	0	1	2	-1	-2
Y=2$-x$	2	1	0	3	4

x	0	1	-1	2	-2
$y = x^2$	0	1	1	4	4



Area = $\iint dy\,dx$

$$y : x^2 \to 2 - x$$

$$x : -2 \to 1$$

$$\int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx = \int_{-2}^{1}[y]_{x^2}^{2-x}\,dx$$

$$= \int_{-2}^{1}(2 - x - x^2)dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left[2 - \frac{1}{2} - \frac{1}{3}\right] - \left[-4 - \frac{4}{2} + \frac{8}{3}\right] = \frac{27}{6}$$

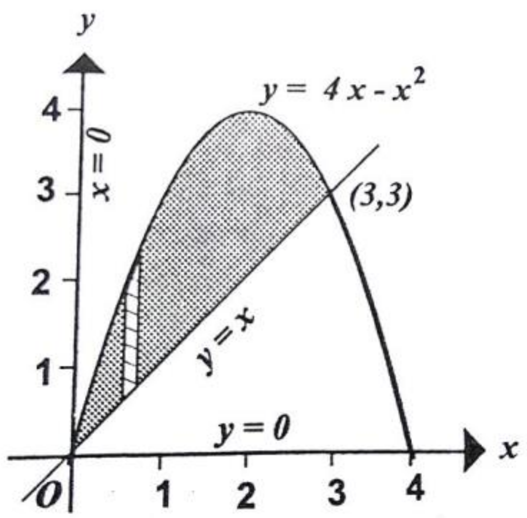
Example: 4.37

Find by double integration the area lying between the parabola y=4x-x^2 and the line y=x

Solution:

Given $y = 4x - x^2$ and $y = x$

x	0	1	2	-1	-2	3
$y = 4x - x^2$	0	3	4	-5	-12	3



$Area = \iint dy\,dx$

$$y : x \to 4x - x^2$$

$$x : 0 \to 3$$

$$\int_0^3\int_x^{4x-x^2} dy\,dx = \int_0^3[y]_x^{4x-x^2}\,dx$$

$$= \int_0^3(4x - x^2 - x)dx$$

$$= \int_0^3(3x - x^2)dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}$$