Double integration in Cartesian co – ordinates

Let $f(x, y)$ be a single valued function and continuous in a region R bounded by a closed curve C. Let the region R be subdivided in any manner into n sub regions $R_1, R_2, R_3, \cdots, R_n$ of areas $A_1, A_2, A_3, \cdots, A_n$.Let (x_i, y_j) be any point in the sub regionR_i. Then consider the sum formed by multiplying the area of each sub – region by the value of the function $f(x, y)$ at any point of the sub – region and adding up the products which we denote

$$\sum_1^n f(x_i, y_j)A_i$$

The limit of this sum (if it exists) as $n \to \infty$ in such a way that each $A_i \to 0$ is defined as the double integral of $f(x, y)$ over the region R. Thus

$$\lim_{n \to \infty} \sum_1^n f(x_i, y_j)A_i = \iint_R f(x, y)\, dA$$

The above integral can be given as

$$\iint_R f(x,y)dydx \quad or \quad \iint_R f(x,y)dxdy$$

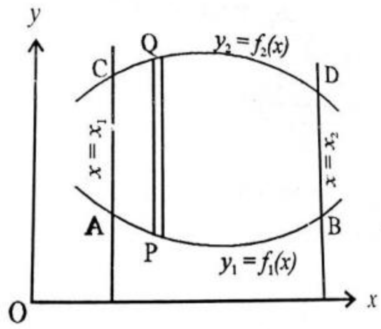
Evaluation of Double Integrals

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dx\,dy$ we first integrate f(x, y) with respect to x partially, that is treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be function of y. Then we integrate this function of with respect to y between the limits y_0 and $y_1$as used.

Region of Integration

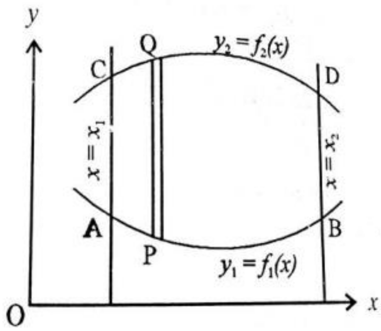
Case (i) Consider the integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dy\,dx$ Given that y varies from $y = f_1(x)$ to $y = f_2(x)$ x varies from $x = a \; to \; x = b$. We get the region R by $y = f_1(x)$,

$y = f_2(x)$, $x = a$, $x = b$. The points A, B, C, D are obtained by solving the intersecting curves. Here the region divided into vertical strips $(dy\, dx)$.



Case (ii) Consider the integral $\int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y) dx\, dy$

Here varies from $x = f_1(y)$ to $x = f_2(y)$ and y varies from $y = c \ to \ y = d$ $\therefore$ the region is bounded by $x = f_1(y)$, $x = f_2(y)$, $y = c$, $y = d$. The points P, Q, R, S are obtained by solving the intersecting curves. Here the region divided into horizontal strips $(dx\, dy)$.



Case (ii) Consider the integral $\int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y) dx\, dy$

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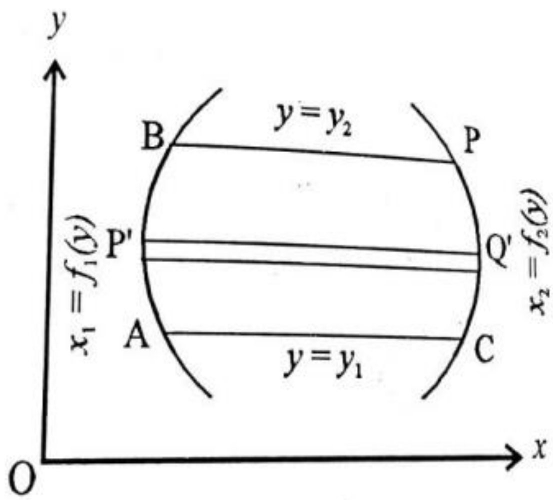

Problems based on Double Integration in Cartesian co-ordinates

Example: 4.1

Evaluate $\int_0^1 \int_1^2 x(x+y)dydx$

Solution:

$$\int_0^1 \int_1^2 x(x+y)dydx = \int_0^1 \int_1^2 (x^2+xy)dydx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \left[(2x^2+2x) - (x^2+\frac{x}{2})\right] dx$$

$$= \int_0^1 \left[2x^2+2x-x^2-\frac{x}{2})\right] dx$$

$$=\int_0^1 \left[x^2 + \frac{3}{2}x\right] dx$$

$$=\left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2}\right]_0^1 \quad =\left(\frac{1}{3}+\frac{3}{4}\right) - (0+0) \quad = \frac{13}{12}$$

Example: 4.2

Evaluate $\int_0^a \int_0^b xy(x-y)dydx$

Solution:

$$\int_0^a \int_0^b xy(x-y)dydx \quad = \int_0^a \int_0^b (x^2y - xy^2)dydx$$

$$=\int_0^a \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^b dx$$

$$=\int_0^a \left[\left(\frac{b^2x^2}{2} - \frac{b^3x}{2}\right) - (0-0)\right] dx$$

$$=\left[\left(\frac{b^2x^3}{6} - \frac{b^3x^2}{6}\right)\right]_0^a$$

$$=\left(\frac{a^3b^2}{6} - \frac{a^2b^3}{6}\right) - (0-0)$$

$$=\frac{a^2b^2}{6}(a-b)$$

Example: 4.3

Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

Solution:

$$\int_2^a \int_2^b \frac{dxdy}{xy} \quad = \int_2^a \left[\frac{1}{y}logx\right]_2^b dy$$

$$=\int_2^a \frac{1}{y}(logb - log2)dy$$

$$= \int_2^a \frac{1}{y}log\left(\frac{b}{2}\right)dy \quad \left[\because log\frac{a}{b} = loga - logb\right]$$

$$=log\frac{b}{2}\int_2^a \frac{1}{y}dy \quad = log\frac{b}{2}[\log y]_2^a$$

$$= log\frac{b}{2}[loga - log2] \quad =\left[log\frac{b}{2}\right]\left[log\frac{a}{2}\right]$$

Example: 4.4

Evaluate $\int_0^1 \int_2^3 (x^2+y^2)\,dx\,dy$

Solution:

$$\int_0^1 \int_2^3 (x^2+y^2)\,dxdy = \int_0^1 \left[\frac{x^3}{3}+y^2x\right]_2^3 dy$$

$$= \int_0^1 \left[\left(\frac{3^3}{3}+3y^2\right)-\left(\frac{2^3}{3}+2y^2\right)\right] dy$$

$$= \int_0^1 \left[9+3y^2-\frac{8}{3}-2y^2\right] dy$$

$$= \int_0^1 \left[\frac{19}{3}+y^2\right] dy \quad = \left[\frac{19y}{3}+\frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{19}{3}+\frac{1}{3}\right] = \frac{20}{3}$$

Example: 4.5

Evaluate $\int_0^3 \int_0^2 e^{x+y}\,dydx$

Solution:

$$\int_0^3 \int_0^2 e^{x+y}\,dydx = \int_0^3 \int_0^2 e^x e^y\,dydx = \left[\int_0^3 e^x\,dx\right]\left[\int_0^2 e^y\,dy\right]$$

$$= [e^x]_0^3[e^y]_0^2 = [e^3-e^0][e^2-e^0]$$

$$= [e^3-1][e^2-1]$$

Example: 4.6

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx = \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left[\sqrt{a^2-x^2}\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\left(0+\frac{a^2}{2}\sin^{-1}1\right) - (0+0)\right] \qquad \left[\because \sin^{-1}1 = \frac{\pi}{2}, \sin^{-1}0 = 0\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

Example: 4.7

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dydx = \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y+y^3)dydx$$

$$= \int_0^a \left[\frac{x^2y^2}{2} + \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{x^2(a^2-x^2)}{2} + \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \int_0^a \left[\frac{a^2x^2}{2} - \frac{x^4}{2} + \frac{a^4}{4} + \frac{x^4}{4} - \frac{2a^2x^2}{4}\right] dx$$

$$= \left[\frac{a^2x^3}{6} - \frac{x^5}{10} + \frac{a^4x}{4} + \frac{x^5}{20} - \frac{2a^2x^3}{12}\right]_0^a$$

$$= \left[\frac{-x^5}{10} + \frac{a^4x}{4} + \frac{x^5}{20}\right]_0^a$$

$$= \left[\frac{-a^5}{10} + \frac{a^5}{4} + \frac{a^5}{20}\right]$$

$$= \frac{a^5}{5}$$

Example: 4.8

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1 \int_x^{\sqrt{x}} xy(x+y)dydx = \int_0^1 \int_x^{\sqrt{x}} (x^2y + xy^2)dydx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(x^2\frac{x}{2} + x\frac{x^{3/2}}{3}\right) - \left(x^2\frac{x^2}{2} + x\frac{x^3}{3}\right)\right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{5}{6}x^4\right] dx$$

$$= \left[\frac{x^4}{8} + \frac{x^{7/2}}{3(7/2)} - \frac{5}{6}\frac{x^5}{5}\right]_0^1$$

$$= \left(\frac{1}{8} + \frac{2}{21} - \frac{1}{6}\right) - (0 + 0 - 0) = \frac{3}{56}$$

Example: 4.9

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{\left(\sqrt{1+x^2}\right)^2 + y^2}$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}(1) - 0\right] dx \qquad \left[\because tan^{-1}(1) = \frac{\pi}{4}\right]$$

$$=\int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} dx \quad = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}} dx \qquad [tan^{-1}(0) = 0]$$

$$= \frac{\pi}{4}\left[log\left[x+\sqrt{1+x^2}\right]\right]_0^1$$

$$=\frac{\pi}{4}\log(1+\sqrt{2})$$

Example: 4.10

Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dydx$

Solution:

The given integral is in correct form

$$\int_0^4 \int_0^{x^2} e^{y/x}\, dydx = \int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$=\int_0^4 \left[\left(\frac{e^x}{1/x}\right) - \left(\frac{1}{1/x}\right)\right] dx$$

$$=\int_0^4 [xe^x - x] dx \;=\int_0^4 x(e^x - 1)\, dx$$

$$=\left[x(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^4 \quad \text{(by Bernoulli's formula)}$$

$$=\left[4(e^4 - 4) - \left(e^4 - \frac{16}{2}\right) - (0 - 1)\right]$$

$$=4e^4 - 16 - e^4 + 8 + 1$$

$$=3e^4 - 7$$