$$= 4\frac{\partial^2}{\partial z}\left\{ f(z)^{p/2} \cdot (p/2)\, f(\bar{z})^{p/2-1} \cdot f'(\bar{z}) \right\}$$

$$= 4\left\{ p/2 \cdot f(z)^{p/2-1} \cdot f'(z) \cdot p/2 \cdot f(\bar{z})^{p/2-1} \cdot f'(\bar{z}) \right\}$$

$$= \frac{4 \times p^2}{4}\left[f(z) \cdot f(\bar{z})\right]^{\frac{p-2}{2}} \cdot \left[f'(z) \cdot \overline{f'(z)}\right]$$

$$= p^2\, |f(z)|^{p-2} \cdot |f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\, |f(z)|^{p-2} \cdot |f'(z)|^2.$$

When $p = 2$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

Construction of Analytic Functions

Milne – Thompson Method

① Construct an analytic function $f(z)$ for which the real part is $e^x \cos y$.

Sol

$$u = e^x \cos y$$

* $u_x = e^x \cos y$

$u_y = -e^x \sin y$

* Put $x = z$; $y = 0$

$u_x = e^z$

$u_y = 0$.

* $f(z) = \int (u_x - i\,u_y)\, dz + c$

$= \int e^z\, dz + c$

$\Rightarrow \boxed{f(z) = e^z + c}$

② Prove that $u = 2x - x^3 + 3xy^2$ is harmonic. Determine its harmonic Conjugate.

$u_{xx} + u_{yy} = 0.$

$\Rightarrow u$ is harmonic.

Put $x = z$; $y = 0$ in u_x & u_y

$u_x = 2 - 3z^2$

$u_y = 0$

$$f(z) = \int (u_x - i u_y)\,dz + c$$

$$= \int (2 - 3z^2)\,dz + c$$

$$= 2z - \frac{3z^3}{3} + c$$

$$\boxed{f(z) = 2z - z^3 + c}$$

$$u + iv = 2(x+iy) - (x+iy)^3 + c$$

$$= 2x + 2iy - x^3 + iy^3 - 3x^2 iy + 3xy^2 + c_1 + ic_2$$

$\Rightarrow u = 2x - x^3 + 3xy^2 + c_1$

(3) Find the analytic function $u+iv$ if $u=(x-y)(x^2+4xy+y^2)$. Also find the conjugate harmonic function v.

Sol)

$$u=(x-y)(x^2+4xy+y^2)$$

$$u=x^3+4x^2y+xy^2-x^2y-4xy^2-y^3$$

* $u_x=3x^2+8xy+y^2-2xy-4y^2$

$u_y=4x^2+2xy-x^2-8xy-3y^2$

* Put $x=z$; $y=0$

$u_x=3z^2$ $\qquad u_y=4z^2-z^2=3z^2$

$\therefore f(z)=\int(u_x-iu_y)\,dz+c$

$$=\int(3z^2-i\,3z^2)\,dz+c$$

$$=3(1-i)\int z^2\,dz+c$$

$u+iv = (1-i)(x+iy)^3 + c$

$= (1-i)\left[x^3 - iy^3 + 3x^2 iy - 3xy^2\right] + c$

$u+iv = x^3 - iy^3 + 3x^2 iy - 3xy^2 - ix^3$

$-y^3 + 3x^2 y + ixy^2 + c_1 + ic_2$

$\therefore\ u = x^3 - 3xy^2 + 3x^2 y - y^3 + c_1$

$v = -y^3 + 3x^2 y - x^3 + xy^2 + c_2.$

(4) Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Also find its conjugate.

Sol

$u = \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{1}{2} \times \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$

$$u_{yy} = \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad (2)$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic

Put $x = z$; $y = 0$.

$$\therefore u_x = \frac{z}{z^2} = \frac{1}{z}$$

$$u_y = 0.$$

$$f(z) = \int (u_x - i u_y)\,dz + c$$

$$= \int \frac{1}{z}\,dz + c$$

$$\boxed{f(z) = \log z + c}$$

$$u + iv = \log\left[r \cdot e^{i\theta}\right] + c$$

$$= \log r + \log e^{i\theta} + c$$

$$= \log r + i\theta + c$$

(5) P.T. $u = e^x[x\cos y - y\sin y]$ is harmonic and hence find the analytic function $u+iv$.

Sol

$$u = e^x[x\cos y - y\sin y]$$

$$u_x = e^x[\cos y] + e^x[x\cos y - y\sin y]$$

$$u_{xx} = e^x\cos y + e^x[\cos y] + e^x[x\cos y - y\sin y]$$

$$u_y = e^x[-x\sin y - y\cos y - \sin y]$$

$$u_{yy} = e^x[-x\cos y + y\sin y - \cos y - \cos y]$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic.

Put $x = z$; $y = 0$ in u_x & u_y

$\therefore u_x = e^z + z\cdot e^z = (1+z)e^z$

$u_y = e^z[\ ]$

$$= \left[(1+z)\ e^{z} - e^{z}\right] + c$$

$$f(z) = z e^{z} + c$$

(6) Given that $u = \dfrac{\sin(2x)}{\cosh(2y) - \cos(2x)}$

Find the analytic function $f(z) = u + iv$.

Sol

$$u = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)}$$

$$u_x = \frac{2\left[\cosh(2y) - \cos(2x)\right]\cos(2x) - 2\sin^2(2x)}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

$$= \frac{2\cosh(2y)\cos(2x) - 2\cos^2(2x) - 2\sin^2(2x)}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

$$u_x = \frac{2\cosh(2y)\cos(2x) - 2}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

Put $x = z$; $y = 0$.

$$u_x = \frac{2\cos(2z) - 2}{[1 - \cos(2z)]^2} = \frac{-2}{1-\cos(2z)}$$

$$= \frac{-2}{2\sin^2(z)} = -\operatorname{cosec}^2 z$$

$$u_y = 0.$$

$$f(z) = \int (u_x - iu_y)\, dz + c'$$

$$= -\int \operatorname{cosec}^2 z \, dz + c'$$

$$f(z) = \cot z + c'$$

(7) Prove that $e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function and the imaginary part.

Sol

$$u = e^{-2xy} \sin(x^2 - y^2)$$

$$u_x = 2x\, e^{-2xy} \cos(x^2 - y^2) - 2y\, e^{-2xy} \sin(x^2 - y^2)$$

$$f(z) = \int (u_x - i u_y)\, dz + c$$

$$= \int \left[2z \cos(z^2) + 2iz \sin(z^2) \right] dz + c$$

$$= 2 \int z \left[\cos(z^2) + i \sin(z^2) \right] dz + c$$

$$= 2 \int z \cdot e^{iz^2}\, dz + c$$

$$\cancel{= 2 \left\{ z \frac{e^{iz^2}}{i2z} - 1\, e^{iz^2} \right\} + c}$$

$$\cancel{= 2x e^{iz^2} \left[\frac{1}{2i} - 1 \right] + c}$$

Take $t = z^2 \Rightarrow dt = 2z\, dz$

$$\Rightarrow z\, dz = \frac{dt}{2}$$

$$= \int e^{it}\, dt + c = \frac{e^{it}}{i} + c$$

$$\boxed{f(z) = -i e^{it} + c} = -i e^{iz^2} + c$$

$$u+iv = -i\, e^{-2xy}\left[\cos(x^2-y^2) + i\sin(x^2-y^2)\right] + c$$

$$\Rightarrow \quad v = -e^{-2xy}\cos(x^2-y^2)$$

Type (2) – Imaginary part v is given

(i) Find V_x and V_y

(ii) Put $x = z$ and $y = 0$.

(iii) $f(z) = \int (V_y + i V_x)\, dz + c$.

① Show that $v = e^{-x}[x\cos y + y\sin y]$ is harmonic function. Hence find its analytic function $f(z) = u+iv$.

Sol

$$V = e^{-x}[x\cos y + y\sin y]$$

$$V_x = e^{-x}\cos y - e^{-x}[x\cos y + y\sin y]$$

$$V_{xx} = -e^{-x}\cos y - e^{-x}\cos y + e^{-x}[x\cos y + y\sin y]$$

$$V_y = e^{-x}[-x\sin y + y\cos y + \sin y]$$

Put $x = z$ & $y = 0$.

$V_x = e^{-z} - e^{-z}.z = (1-z)e^{-z}$

$V_y = 0.$

$f(z) = \int (V_y + iV_x)\, dz + c$

$= \int (1-z)e^{-z} dz + c$

$= \left[(1-z)\dfrac{e^{-z}}{-1} + e^{-z}\right] + c$

$f(z) = -(1-z)e^{-z} + e^{-z} + c$

(2) Can $v = \tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function $f(z) = u + iv$, taking v as the imaginary part and hence find u.

Sol

Condition to Imaginary part : v is harmonic.

$$V_{xx} = +y\,(x^2+y^2)^{-2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$V_y = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\times\frac{1}{x}$$

$$V_y = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$V_{yy} = -x(x^2+y^2)^{-2}\cdot 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$V_{xx} + V_{yy} = 0 \Rightarrow V$ is harmonic.

Put $x = z$; $y = 0$.

$$V_x = 0 \quad ; \quad V_y = \frac{z}{z^2+0} = \frac{z}{z^2} = \frac{1}{z}$$

$$f(z) = \int (V_y + iV_x)\,dz + c$$

$$= \int \frac{1}{z}\,dz + c$$

$$f(z) = \log z + c$$

Put $z = re^{i\theta}$

$u = \log r = \frac{1}{2}\log(x^2+y^2)$

$\Rightarrow v = \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

(3) If $f(z) = u+iv$ is an analytic function, find $f(z)$ if $v = \log(x^2+y^2) + x - 2y$.

Sol

$$v = \log(x^2+y^2) + x - 2y$$

$$V_x = \frac{2x}{x^2+y^2} + 1 \qquad V_y = \frac{2y}{x^2+y^2} - 2$$

Put $x = z$, $y = 0$

$$V_x = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1 \qquad V_y = -2$$

$$f(z) = \int (V_y + iV_x)\, dz + c$$

$$= \int \left(-2 + \frac{i2}{z} + i\right) dz + c$$

$$f(z) = -2z + 2i\log z + iz + c$$

Type (3) – $u-v$ is given

* $f(z) = \dfrac{F(z)}{1+i}$

~~Method~~

Type (4) – $u+v$ is given

Take $V = u+v$

* Find V_x and V_y

* Put $x = z$; $y = 0$.

* $F(z) = \int (V_y + i V_x)\, dz + c$

* $f(z) = \dfrac{F(z)}{1-i}$

(1) Find the analytic function $f(z) = P + iQ$

if $P - Q = \dfrac{\sin(2x)}{\cosh(2y) + \cos(2x)}$.

Sol

Take $U = \dfrac{\sin(2x)}{\cosh(2y) + \cos(2x)}$

$$U_y = -2\sin(2x)\left[\cosh(2y)+\cos(2x)\right]^{-2}\cdot \sinh(2y)$$

$$U_y = \frac{-2\sin(2x)\sinh(2y)}{\left[\cosh(2y)+\cos(2x)\right]^2}$$

Put $x = z\ ;\ y = 0$

$$U_x = \frac{2\cos(2z)+2}{\left[1+\cos(2z)\right]^2} = \frac{2}{1+\cos(2z)}$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z$$

$$U_y = 0$$

$$F(z) = \int (U_x - iU_y)\,dz + c$$

$$= \int \sec^2 z\,dz + c$$

(2) Determine the analytic function $f(z)=u+iv$ given $u-v=\dfrac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}$ and $f\left(\frac{\pi}{2}\right)=0$.

Sol

Take $U=\dfrac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}$

$$U_x=\frac{2(\cos x-\cosh y)\left[-\sin x+\cos x\right]+\left[\cos x+\sin x-e^{-y}\right].2\sin x}{4(\cos x-\cosh y)^2}$$

$$U_y=\frac{2(\cos x-\cosh y).e^{-y}+2\left[\cos x+\sin x-e^{-y}\right]\sinh y}{4(\cos x)-\cosh y)^2}$$

Put $x=z$; $y=0$.

$$U_x=\frac{2(\cos z-1)\left[-\sin z+\cos z\right]+2\left[\cos z+\sin z-1\right]\sin z}{4(\cos z-1)^2}$$

$$= \frac{2(1-\cos z)}{4(\cos z - 1)^2} = \frac{1}{2\times 2\sin^2\left(\frac{z}{2}\right)} = \frac{1}{4\sin^2\left(\frac{z}{2}\right)}$$

$$U_y = \frac{2(\cos z - 1) + 2[\cos z + \sin z - 1]\times 0}{4(\cos z - 1)^2}$$

$$= \frac{2(\cos z - 1)}{4(\cos z - 1)^2} = \frac{-2(1-\cos z)}{4(1-\cos z)^2}$$

$$= \frac{-1}{2(1-\cos z)} = \frac{-1}{2\times 2\sin^2\left(\frac{z}{2}\right)}$$

$$U_y = \frac{-1}{4\sin^2\left(\frac{z}{2}\right)} = \frac{-1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$F(z) = \int (U_x - iU_y)\,dz + c'$$

$$= \int \left[\frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right) + \frac{i}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)\right] dz + c'$$

$$= \frac{1}{4}(1+i)\int \operatorname{cosec}^2\left(\frac{z}{2}\right) dz + c'$$

$$f(z) = \frac{F(z)}{1+i}$$

$$f(z) = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + \frac{c}{1+i} = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + c_1$$

Take $z = \pi/2$

$$0 = \frac{-1}{2}\cot\left(\frac{\pi}{4}\right) + c_1$$

$$= \frac{-1}{2} + c_1 \Rightarrow \boxed{c_1 = \frac{1}{2}}$$

$$\therefore f(z) = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + \frac{1}{2}.$$

(3) Find the analytic function $f(z) = u+iv$ if $u+v = e^x[\cos y + \sin y]$

Sol

$$V = e^x[\cos y + \sin y]$$

$$V_x = e^x[\cos y + \sin y]$$

$$V_y = e^x[-\sin y + \cos y]$$

$= (1+i)\, e^{z} + c$

$\Rightarrow f(z) = \dfrac{F(z)}{1+i}$

$\Rightarrow f(z) = e^{z} + \dfrac{c}{1+i}$

Transformation

A complex valued function of complex variable $w = f(z)$ can be treated as a transformation of points of Z-plane into points of W-plane.

Invariant or Fixed point

The invariant (or) fixed points of the transformation $w = f(z)$ is given by solving the equation $z = f(z)$.

(1) Find the invariant points of z^2.

Sol