3.6. BILINEAR TRANSFORMATION

- **3.5.a. Introduction**

The transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

This transformation was first introduced by A.F. Mobius, So it is also called Mobius transformation.

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

The bilinear transformation which transforms z_1, z_2, z_3, into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az + b) = 0$

It is linear both in w *and* z that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$$i.e., \frac{ad - bc}{(cz + d)^2} \neq 0$$

$$i.e., ad - bc \neq 0$$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z-z_1}{z_2-z_1}$ by 1 (or)Omit the factors involving ∞

Example: 3.59 Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing w by z

$$z = \frac{2zi+5}{z-4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36+20}}{2} \qquad \therefore z = 5i, i$$

Example: 3.60 Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing w by z

$$z = \frac{1+z}{1-z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: 3.61 Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$. [Anna, May 1996]

Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\,; \qquad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Example: 3.62 Find the fixed point of the transformation $w = \frac{6z-9}{z}$. [A.U N/D 2005]

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3,3$$

The fixed points are $3, 3$.

Example: 3.63 Find the invariant points of the transformation $w = \frac{2z+6}{z+7}$. [A.U M/J 2009]

Solution:

The invariant (fixed) points are given by

$$w = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$$\Rightarrow z = -6, z = 1$$

Example: 3.64 Find the invariant points of $f(z) = z^2$. [A.U M/J 2014 R-13]

Solution:

The invariant points are given by $z = w = f(z)$

$$\Rightarrow z = z^2$$

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\Rightarrow z = 0, \quad z = 1$$

Example 3.65 Find the invariant points of a function $f(z) = \frac{z^3+7z}{7-6zi}$. [A.U D15/J16 R-13]

Solution:

Given $w = f(z) = \frac{z^3+7z}{7-6zi}$

The invariant points are given by

$$\Rightarrow z = \frac{z^3+7z}{7-6zi}$$

$$\Rightarrow 7 - 6zi = z^2 + 7$$

$$\Rightarrow -6zi = z^2 \Rightarrow z^2 + 6zi = 0 \Rightarrow z(z + 6i) = 0$$

$$\Rightarrow z = 0,\ z = -6i$$

PROBLEMS BASED ON BILINEAR TRANSFORMATION

Example: 3.66 Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively. [A.U. A/M 2015 R-13, A.U N/D 2013, N/D 2014]

Solution:

Given $z_1 = 0,\ z_2 = -1,\ z_3 = i,$

$w_1 = i,\ w_2 = 0,\ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z}{(z-i)}(1+i)$$

$$\Rightarrow w - i = \frac{z}{(z-i)}(-i+1)$$

$$\Rightarrow w = \frac{z}{(z-i)}(-i+1) + i = \frac{-iz+z+iz+1}{(z-i)} = \frac{z+1}{z-i}$$

Example: 3.67 Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively. [Anna, May 1997] [A.U N/D 2012] [A.U A/M 2017 R-08]

Solution:

Given $z_1 = \infty,\ z_2 = i,\ z_3 = 0, w_1 = 0,\ w_2 = i,\ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example: 3.68 Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z-$ plane onto the upper half of the $w-$ plane. [A.U. May 2001] [A.U M/J 2014] [A.U D15/J16 R-13]

Solution:

Given $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 0,\ w_2 = 1,\ \ w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \qquad \because \left[\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)\right] = \left[\frac{i^2+i+i+1}{i^2-i^2}\right] = \left[\frac{2i}{-2}\right] = -i$$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1}\left[\because w = \frac{az+b}{cz+d}, ad-bc \neq 0 \text{ Form}\right]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w+i] = -w + i$$

$$\Rightarrow z = \frac{(w-i)}{w+i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow \left|\frac{-(w-i)}{w+i}\right| < 1$$

$$\Rightarrow \left|\frac{w-i}{w+i}\right| < 1$$

$$\Rightarrow |w-i| < |w+i|$$

$$\Rightarrow |u+iv-i| < |u+iv+i|$$

$$\Rightarrow |u+i(v-1)| < |u+i(v+i)|$$

$$\Rightarrow u^2 + (v-1)^2 < u^2 + (v+1)^2$$

$$\Rightarrow (v-1)^2 < (v+1)^2$$

$$\Rightarrow \ v^2 - 2v + 1 < v^2 + 2v + 1$$

$$\Rightarrow \ -4v < 0$$

$$\Rightarrow v > 0$$

Example: 3.69 Determine the bilinear transformation that maps the points $-1, 0, 1$, in the z plane onto the points $0, i, 3i$ in the w plane. [Anna, May 1999]

Solution:

Given $z_1 = -1, \quad z_2 = 0, \quad z_3 = 1,$

$w_1 = 0, \quad w_2 = i, \quad w_3 = 3i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{[z-(-1)][0-1]}{(z-1)[0-(-1)]}$$

$$\Rightarrow \frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\Rightarrow \frac{-2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow \frac{-2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow \frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow 2wz - 2w = wz + w - 3zi - 3i$$

$$\Rightarrow 2wz - 2w - wz - w = -3i(z+1)$$

$$\Rightarrow w[2z - 2 - z - 1] = -3i(z+1)$$

$$\Rightarrow w[z-3] = -3i(z+1)$$

$$\Rightarrow w = -3i\frac{(z+1)}{(z-3)}$$

Note: Either image or object or both are infinity should not apply the following Aliter method.

Example: 3.70 Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, 1, -i$ respectively. **[Anna, May 2002]**

Solution:

Given $z_1 = -1, \ \ z_2 = 0, \ \ z_3 = 2,$

$w_1 = 0, \ \ w_2 = i, \ \ w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Let } A = \frac{w_2-w_3}{w_2-w_1} = \frac{i+i}{i-0} = \frac{2i}{i} = 2$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-2}{0+2} = -1$$

$\Rightarrow a = Aw_1 - Bw_3 = (2)(0) - (-1)(-1) = -i$

$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(-i)(-2) - (2)(0)(2) = -2i$

$\Rightarrow c = A - B = 2 - (-1) = 3$

$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (2)(2) = -2$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i)z+(-2i)}{3z+(-2)}$$